Flow control

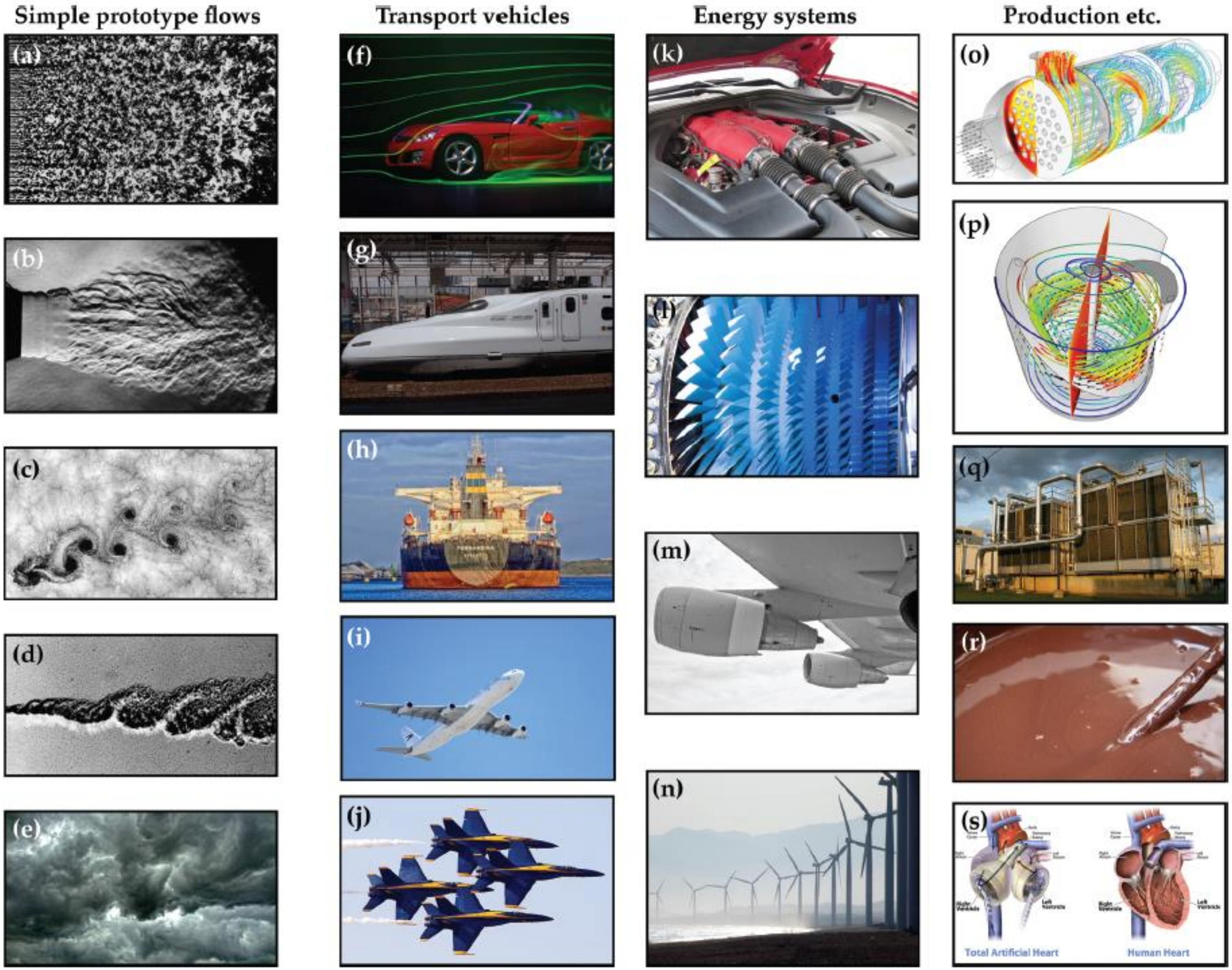
From open-loop to closed-loop control

Laurent CORDIER

Laurent.Cordier@univ-poitiers.fr

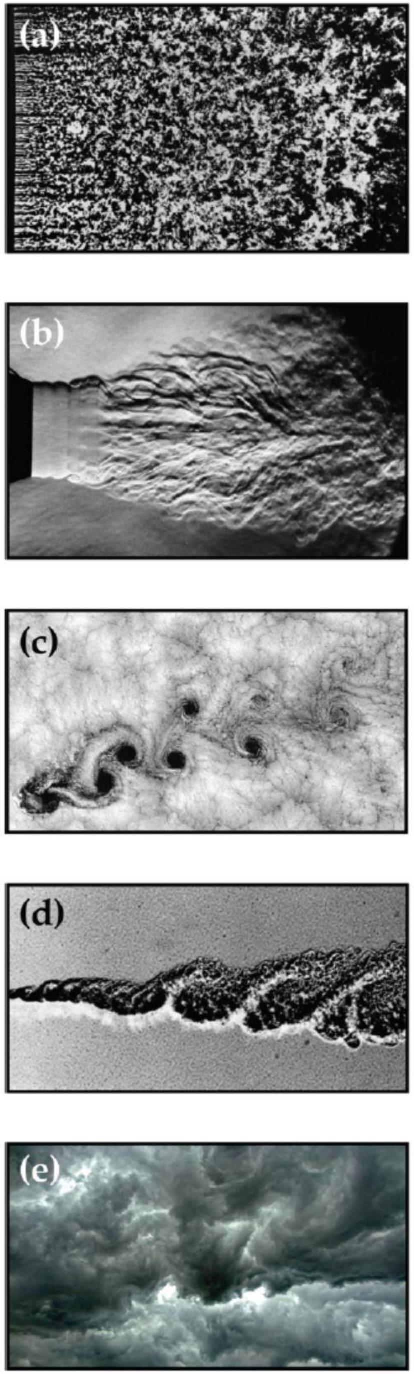
PPRIME Institute

Poitiers, France

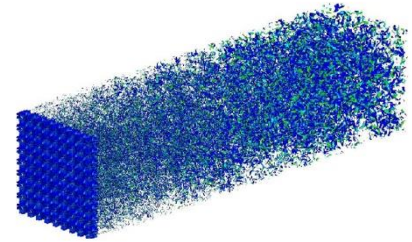


From Brunton, Noack, AMR, 2015

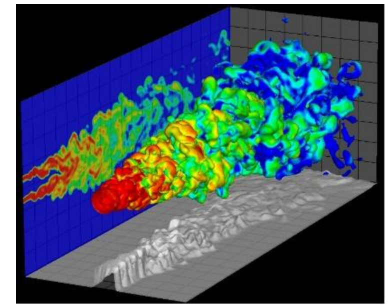
Simple prototype flows



Homogeneous grid turbulence

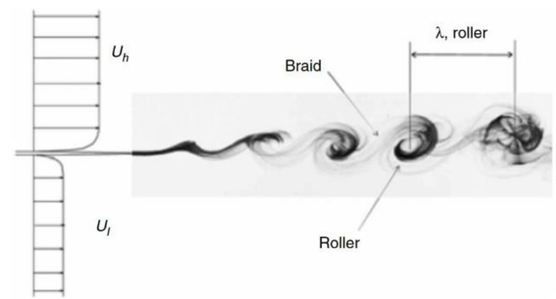


Turbulent jet

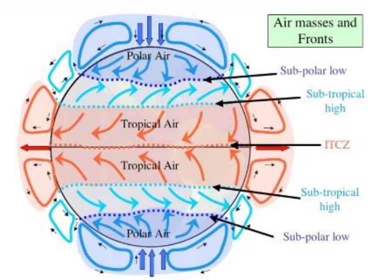


Karman vortex street

Mixing layer



Atmospheric flows: Thunderstorm



Transport vehicles

Car aerodynamics

High-speed train

Cargo ship

Passenger jet

Blue Angel fighter jets

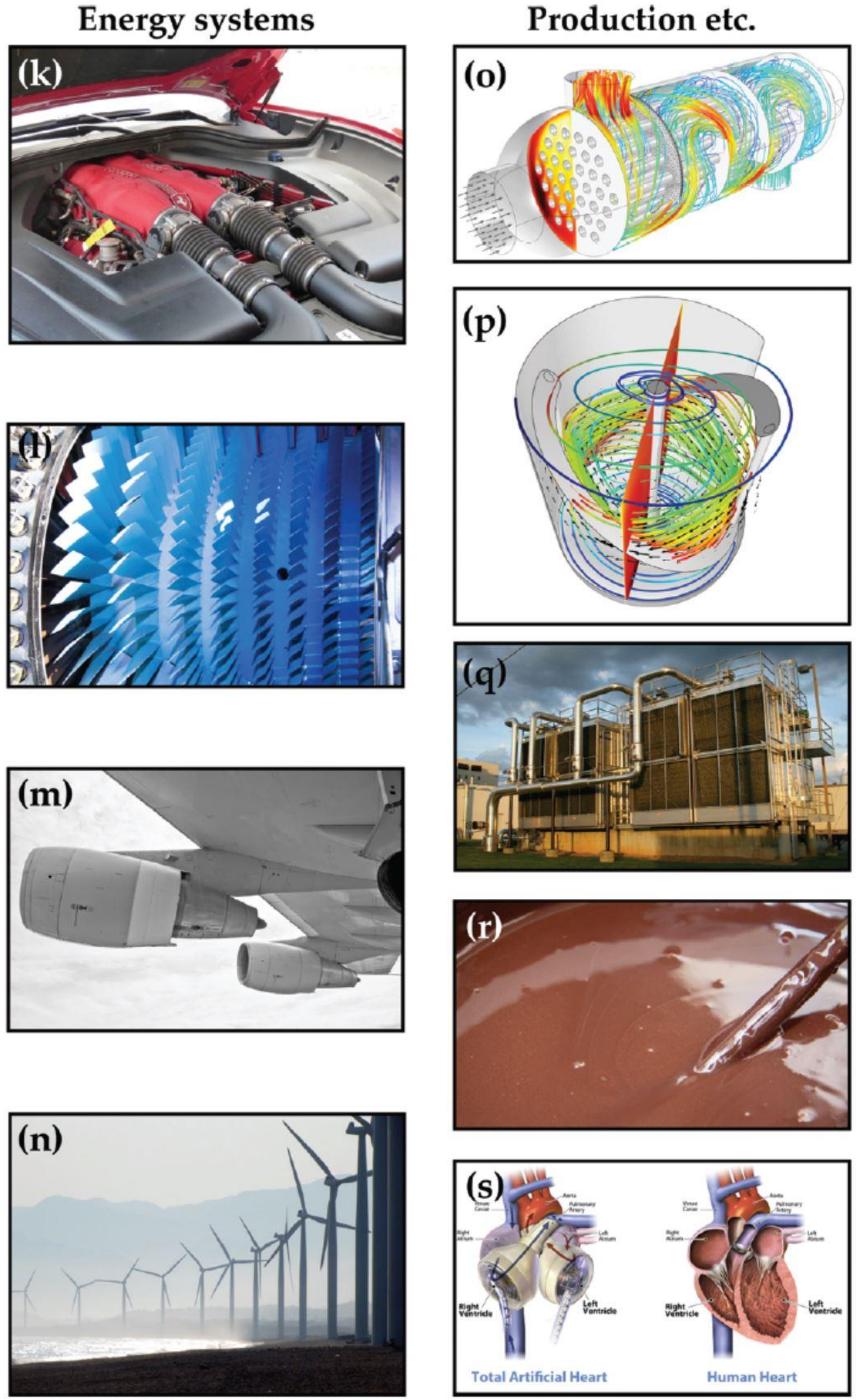
- Drag reduction
- Drag/Lift improvement
- Noise reduction
- Maneuverability
- …….

Automobile engine

Turbo jet engine

Aircraft engines

Wind turbines



Heat exchanger

Mixer

Air conditioner

Chocolate mixing

Artificial heart

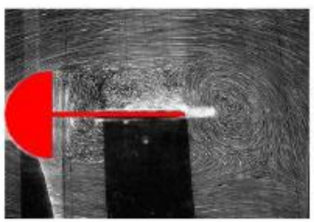
Flow control

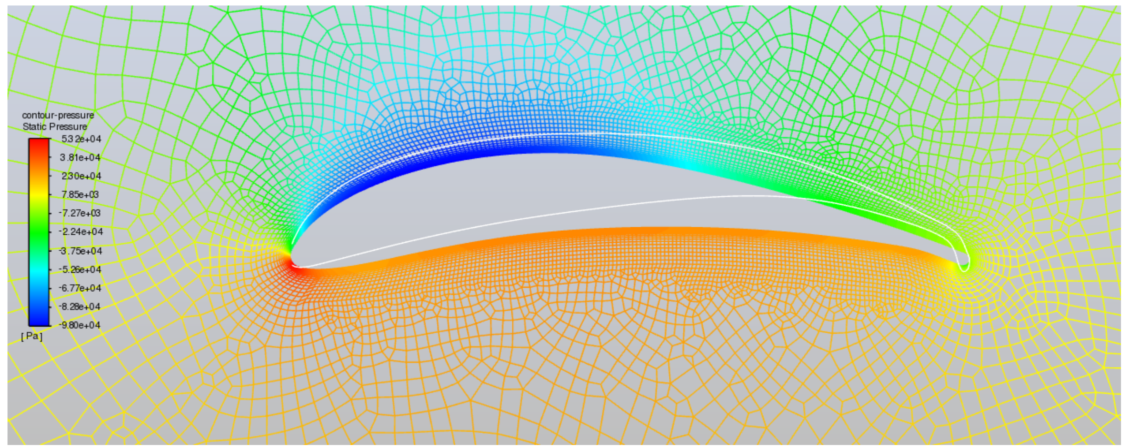
- Introduction of small perturbations (order ϵ) to create order 1 modifications to some flow properties.

Use (if possible) the natural flow instability

Passive

no energy required





Wing shape optimization

Lift maximization

Passive control may be viewed as "shape optimisation"

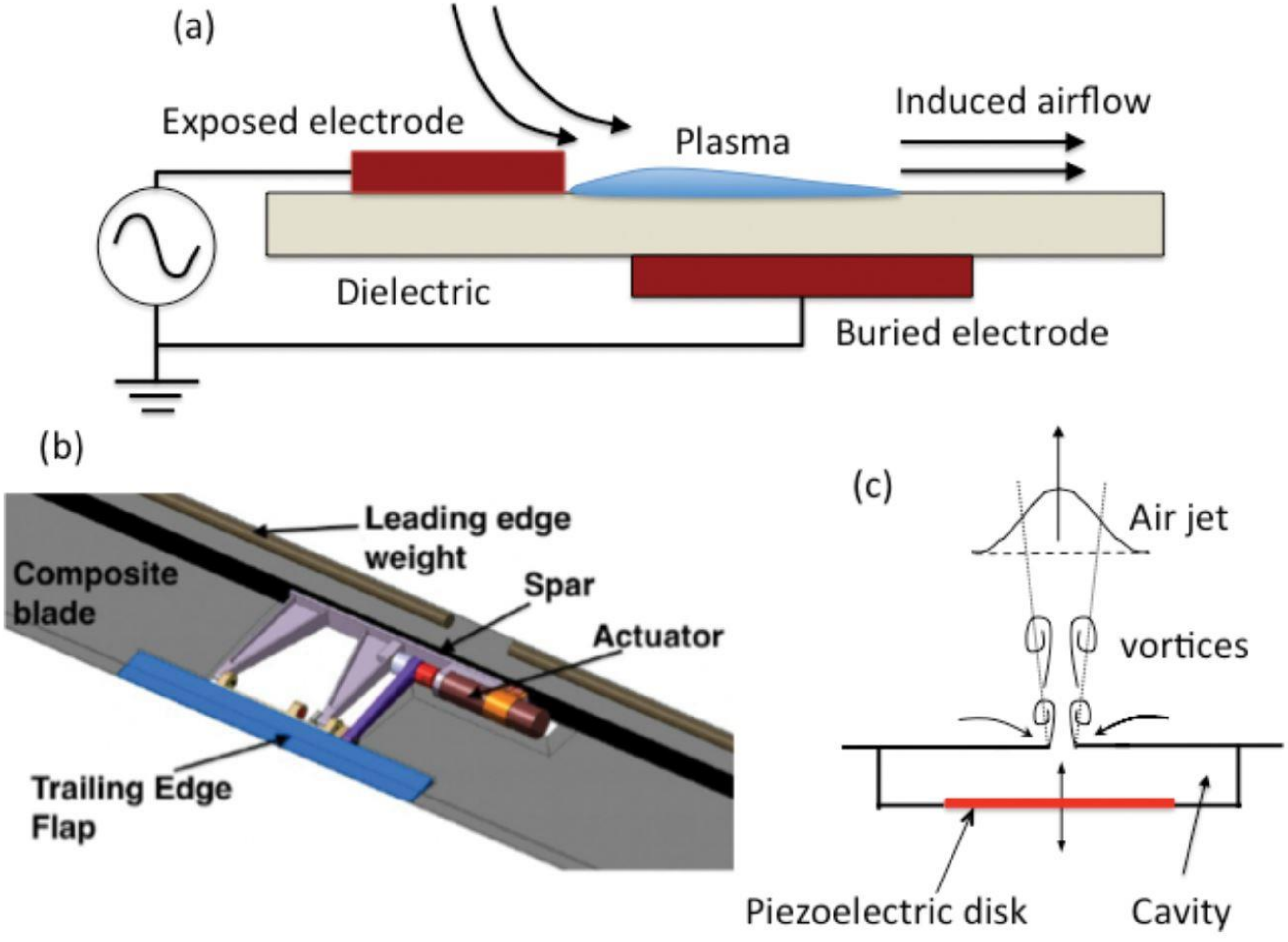
Noise reduction :

Aircraft engine

Sonic boom :

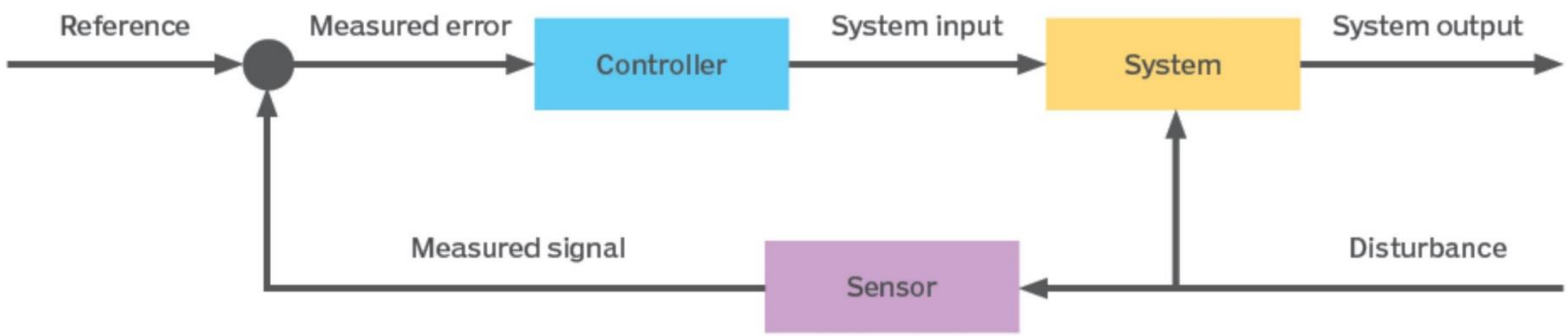
Active

energy required, open loop

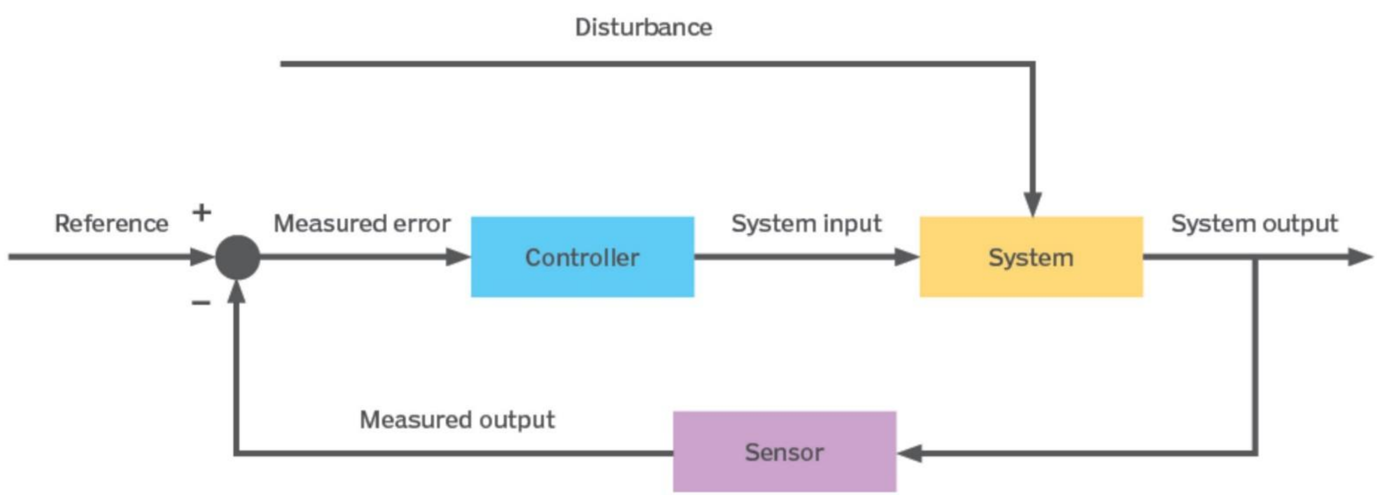


a) Plasma actuator, b) Trailing edge flap, c) Piezoelectric synthetic jet actuator.

Open loop controller



Closed loop controller



Reynolds number → control parameter

$$Re = \frac{V L_c}{\nu}$$

Characteristic length

Characteristic velocity

Kinematic viscosity

Von Karman vortex street

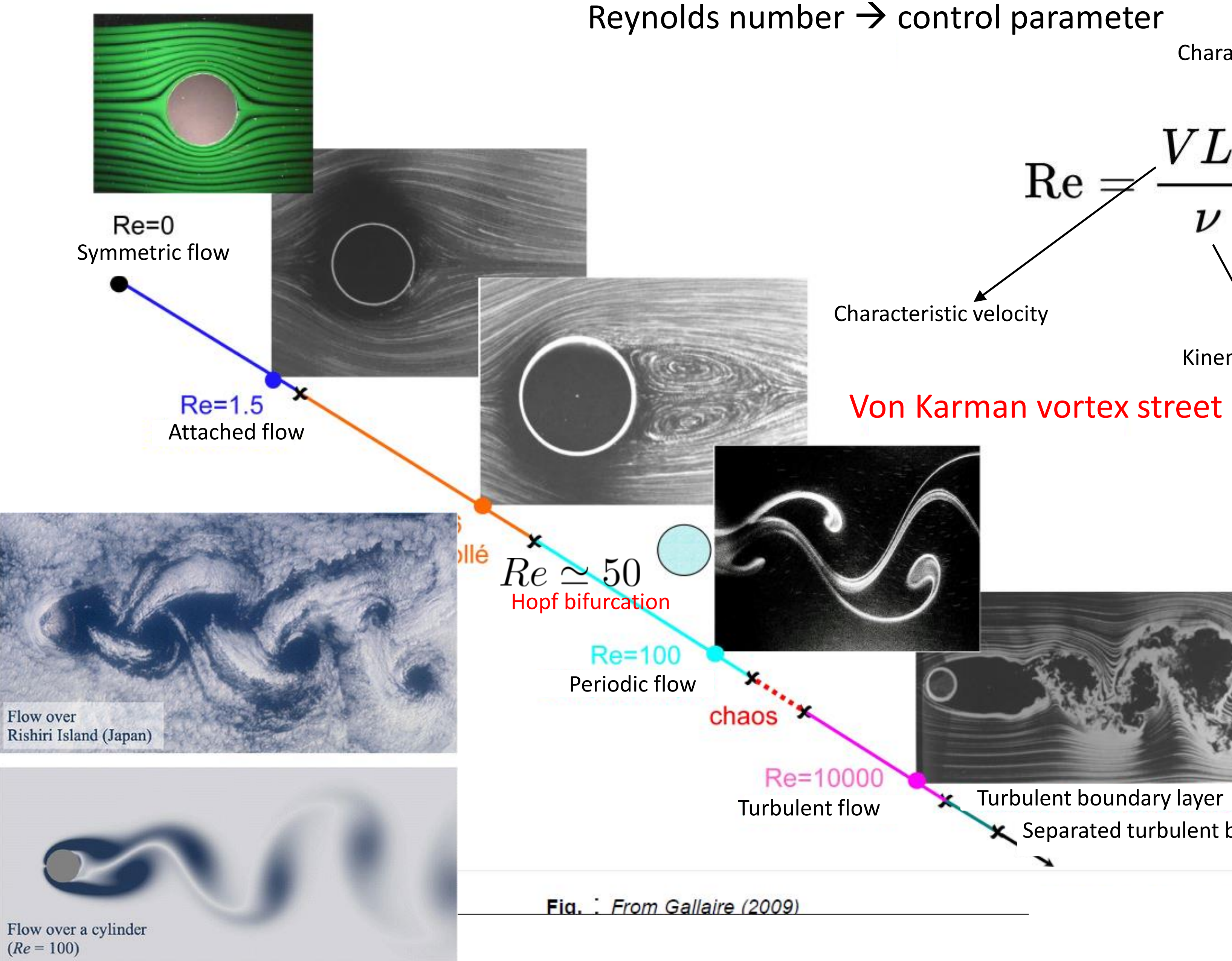


Fig. : *From Gallaire (2009)*

Linear stability analysis (1)

- Navier-Stokes equations (NSE)

$$\begin{aligned}\partial_t \mathbf{U} &= -(\mathbf{U}\cdot\nabla)\mathbf{U} - \nabla P + \frac{1}{R}\nabla^2\mathbf{U} \\ \nabla\cdot\mathbf{U} &= 0 \\ \mathbf{U} &= \mathbf{U}_b \qquad \text{on a boundary set B}\end{aligned}$$

- Base flow (steady solutions to NSE)

$$0 = -(\mathbf{U}\cdot\nabla)\mathbf{U} - \nabla P + \frac{1}{R}\nabla^2\mathbf{U}$$

Linear stability analysis (2)

- Linearization about this base flow

$$
\begin{aligned}
\partial_t \mathbf{u} &= -(\mathbf{U}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{U} - \nabla p + \frac{1}{R}\nabla^2\mathbf{u} \\
\nabla\cdot\mathbf{u} &= 0 \\
\mathbf{u} &= 0 \qquad \text{on a boundary set B}
\end{aligned}
$$

- Search $\mathbf{u}$ in the form $\mathbf{u}(\mathbf{x},t) = \exp(\lambda t)\tilde{\mathbf{u}}(\mathbf{x})$

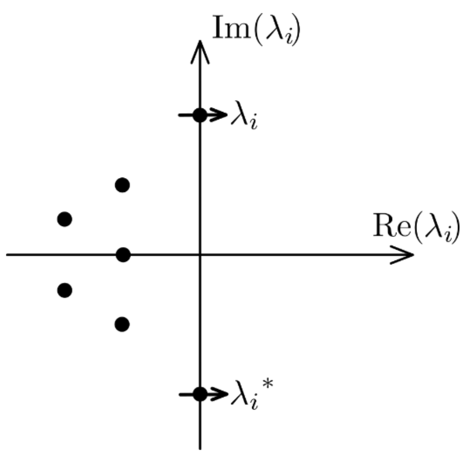
with $\lambda = \sigma + i\omega$

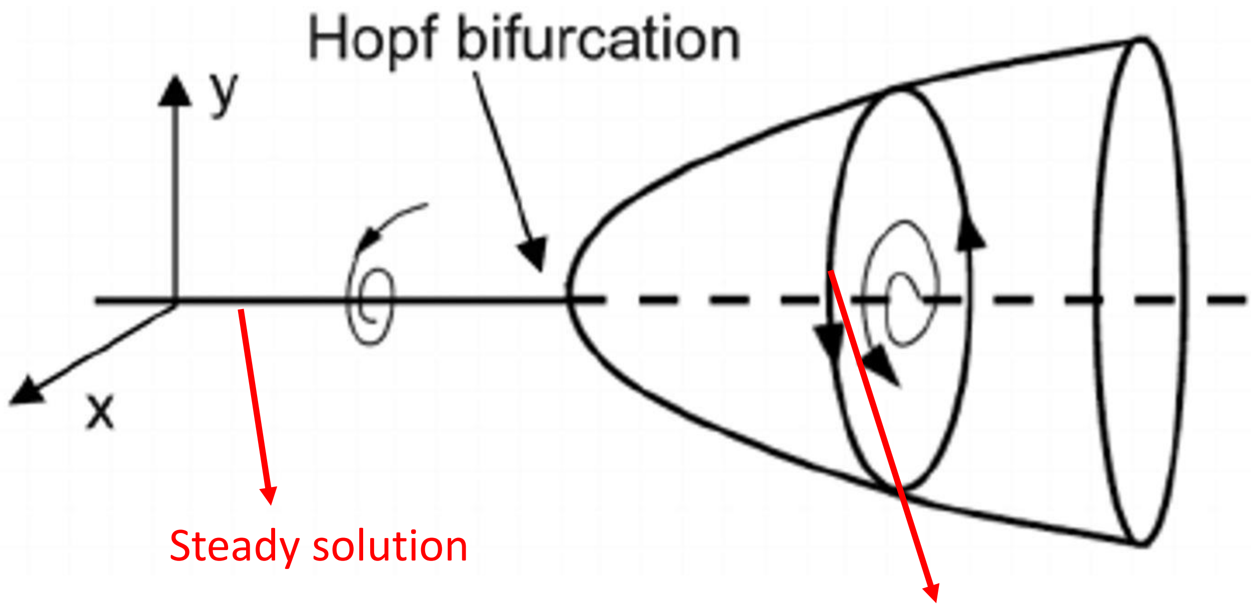
σ : growth rate

ω : frequency

Linear stability analysis (3)

$$\lambda\tilde{\mathbf{u}} = -(\mathbf{U}\cdot\nabla)\tilde{\mathbf{u}} - (\tilde{\mathbf{u}}\cdot\nabla)\mathbf{U} - \nabla p + \frac{1}{R}\nabla^2\tilde{\mathbf{u}}$$





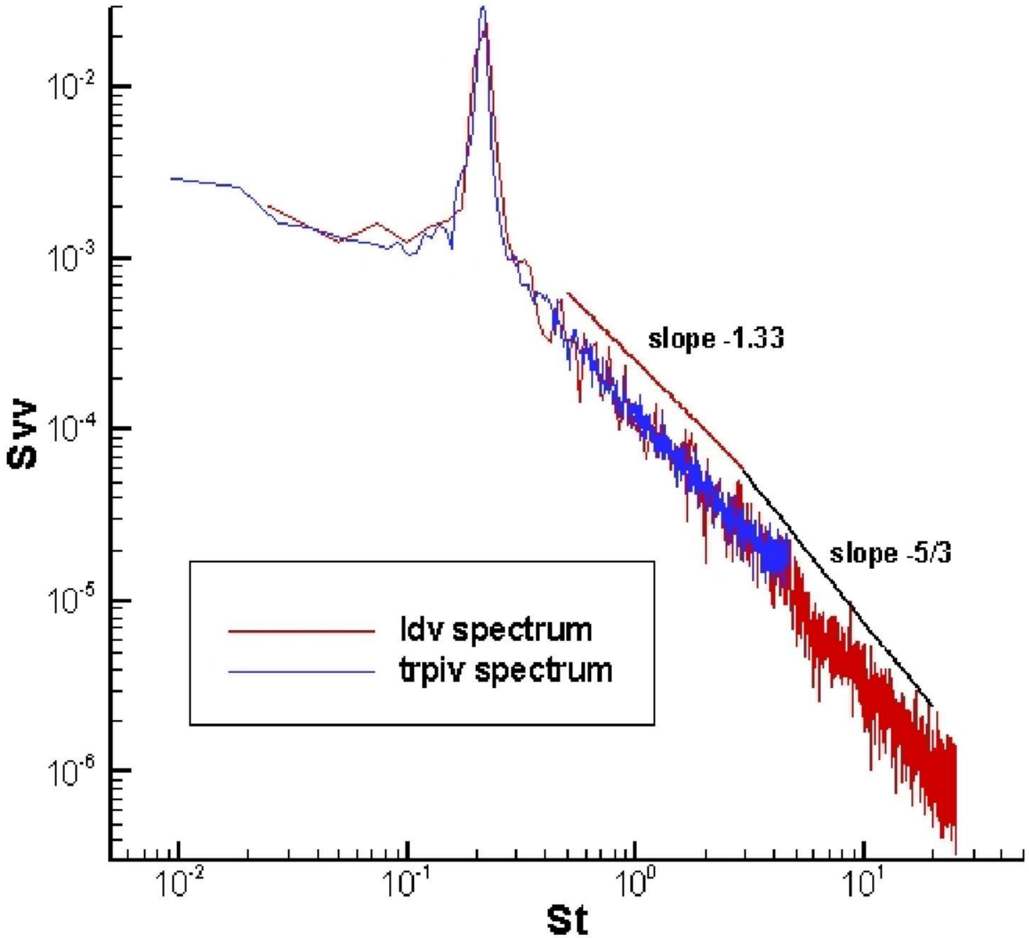
Unsteady solution of period $T = \dfrac{2\pi}{\omega}$

Stable solution

Unstable solution

Hopf bifurcation:
Two complex conjugate eigenvalues cross the imaginary axis.

Velocity spectrum cylinder wake (Re=14000)



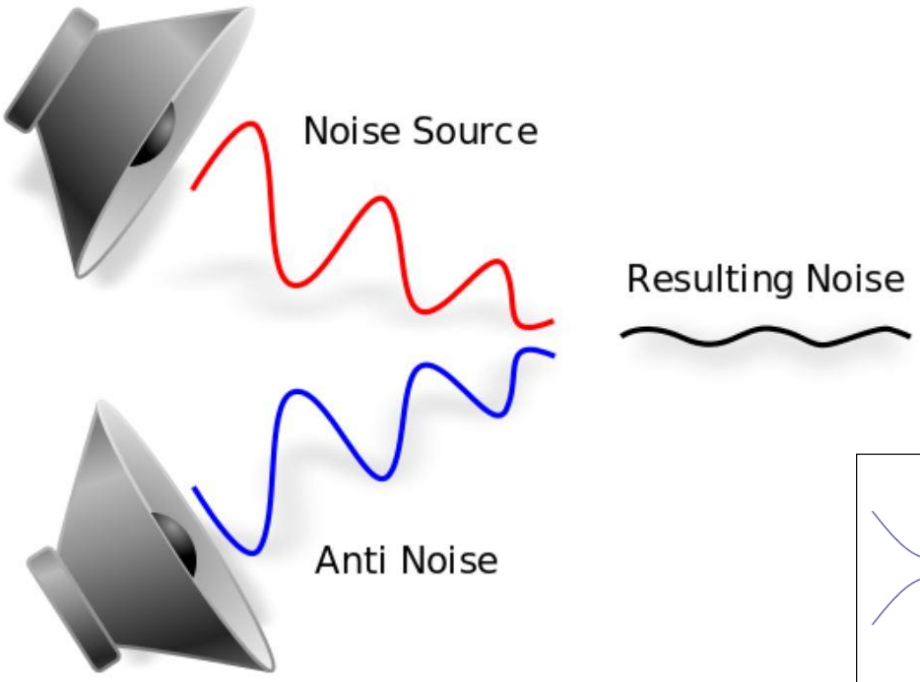
Power spectrum

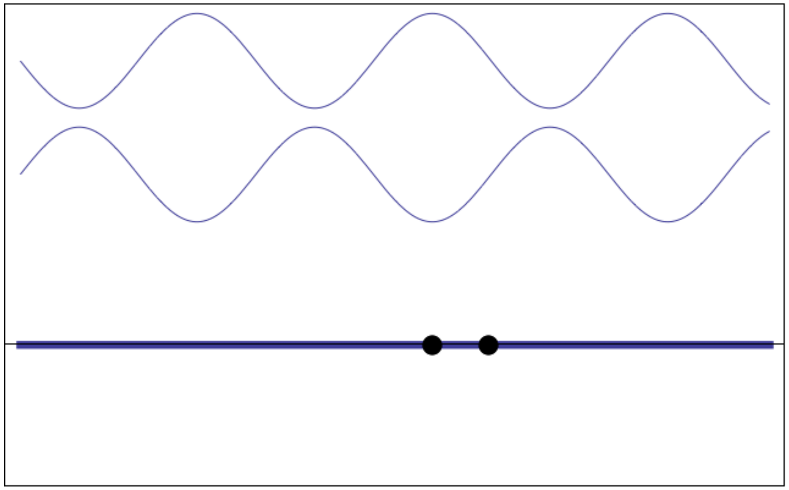
$$S_{xx}(f)$$

Strouhal number

$$\mathrm{St} = \frac{fL}{U}$$

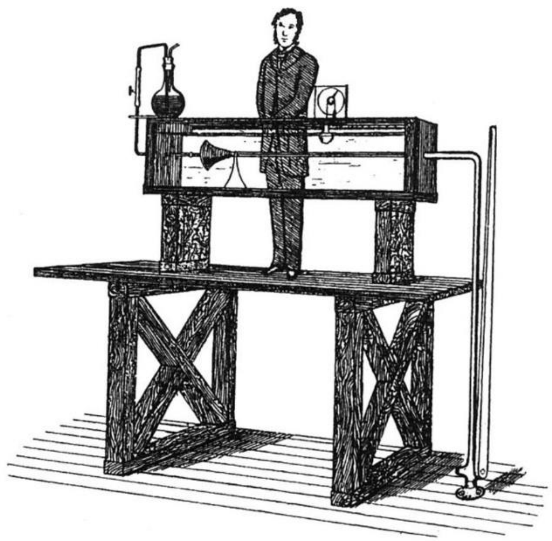
Active noise control

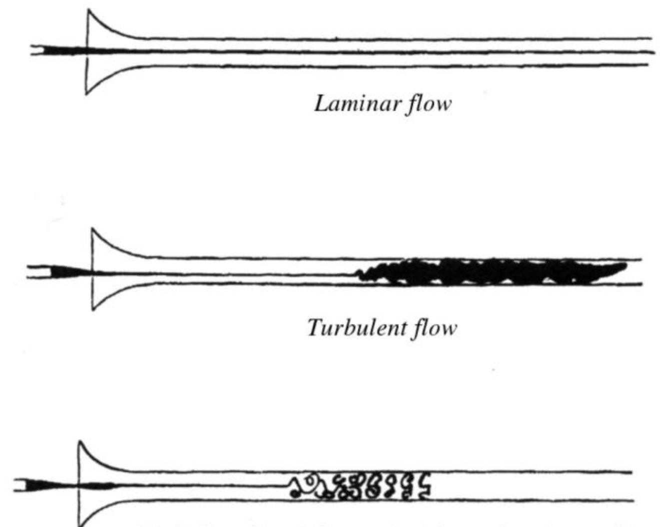




Linear superposition

Laminar Turbulent transition





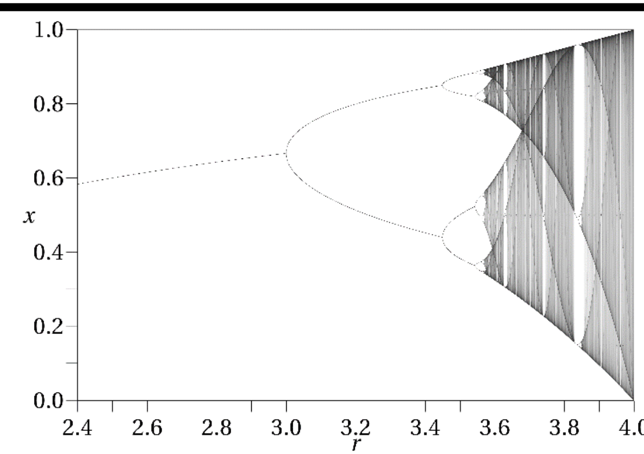
Experimental setup of O. Reynolds (1883) and observations.

Logistic map

$$x_{n+1} = r x_n (1 - x_n)$$

Transition to chaos with a sequence of bifurcations.

Scenario known as « period doubling bifurcation »



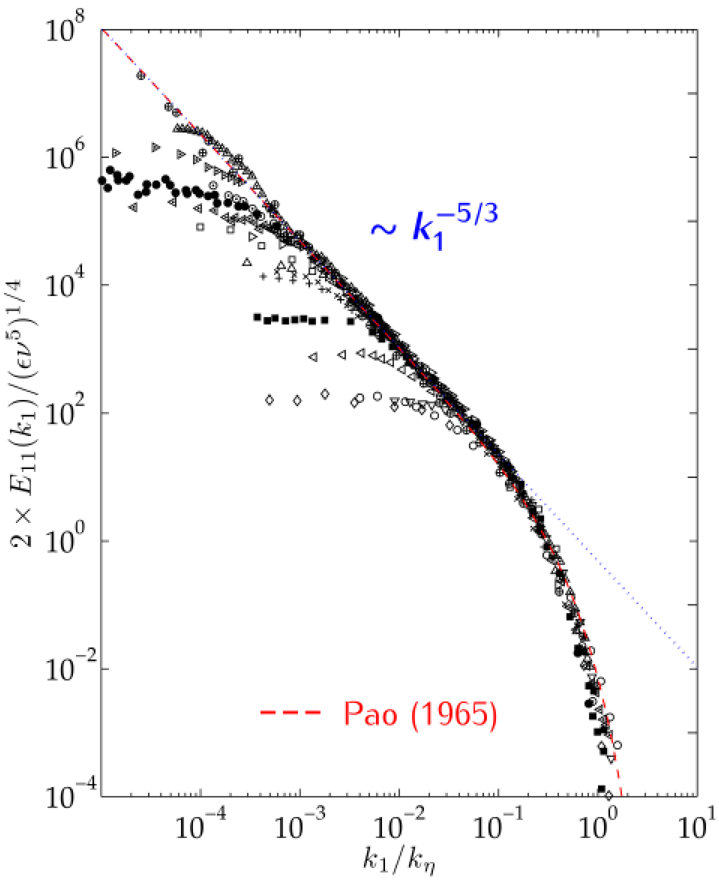
$r \approx 3.56995$

Feedback flow control. Why is it hard?

- Theory of Kolmogorov (energy cascade)

Energy contained in the scale K

$$E_{ij}(\vec{K}) = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} e^{-i\vec{K}\cdot\vec{x}} R_{ij}(\vec{x})\, d\vec{x}$$



Re_{λ_g}

- ○ 23 couche limite (Tielman, 1967)
- ◇ 23 sillage de cylindre (Uberoi & Freymuth, 1969)
- ▽ 37 turbulence de grille (Comte Bellot & Corrsin, 1971)
- ◁ 72 turbulence de grille (Comte Bellot & Corrsin, 1971)
- ■ 130 écoulement cisaillé homogène (Champagne *et al.*, 1970)
- \+ 170 écoulement de conduite (Laufer, 1952)
- × 282 couche limite (Tielman, 1969)
- □ 308 sillage de cylindre (Uberoi & Freymuth, 1969)
- △ 401 couche limite (Sanborn & Marshall, 1965)
- ▷ 540 turbulence de grille (Kistler & Vrebalovich, 1966)
- ◁ 600 couche limite (Saddoughi, 1994)
- ⊙ 780 jet circulaire (Gibson, 1963)
- ● 850 couche limite (Coantic & Favre, 1974)
- ▷ 1500 couche limite (Saddoughi, 1994)
- ⊕ 2000 bras de mer (Grant *et al.*, 1962)
- △ 3180 canal à retour (CAHI Moscou, 1991)

Fig. : *From Bailly et Comte-Bellot (2003)*

Feedback flow control. Why is it hard?

Turbulence viewpoint

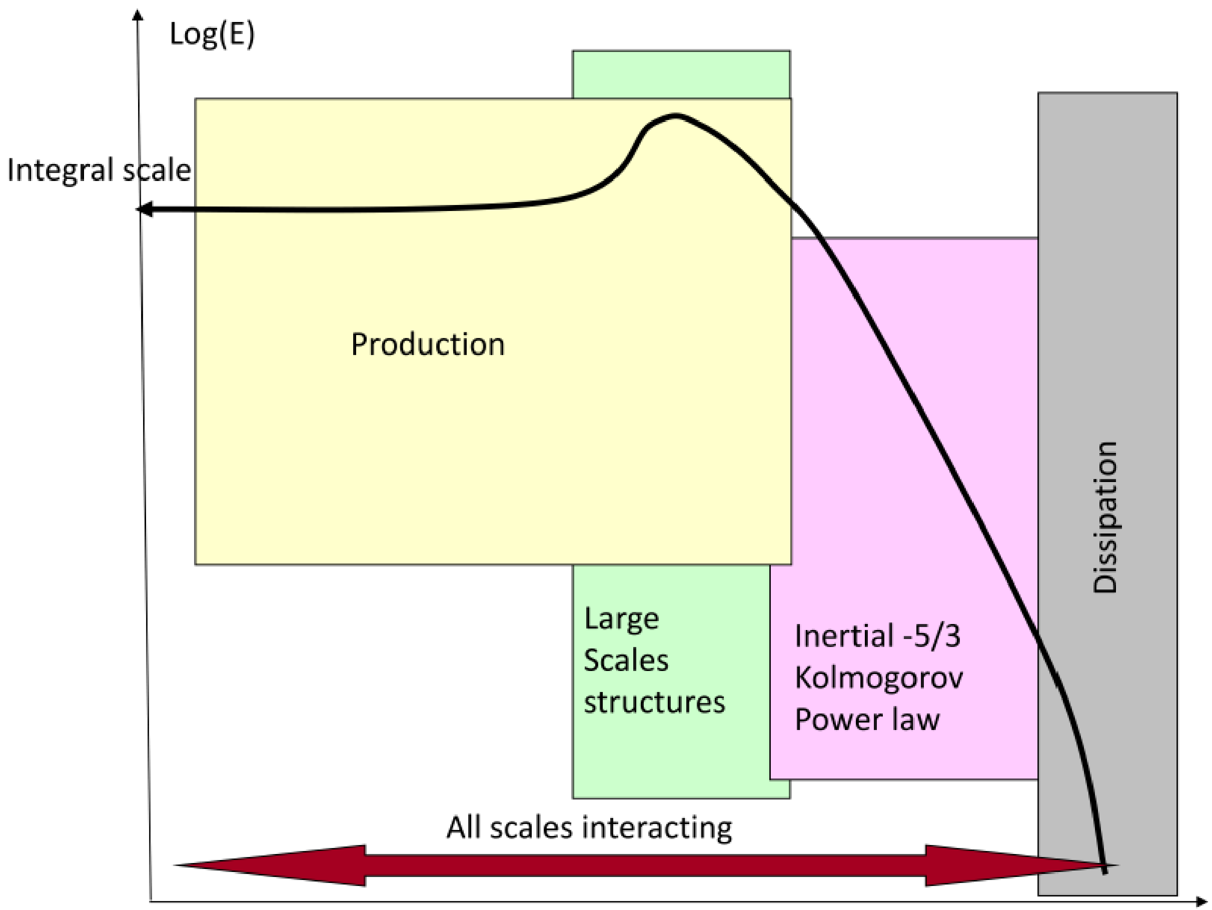


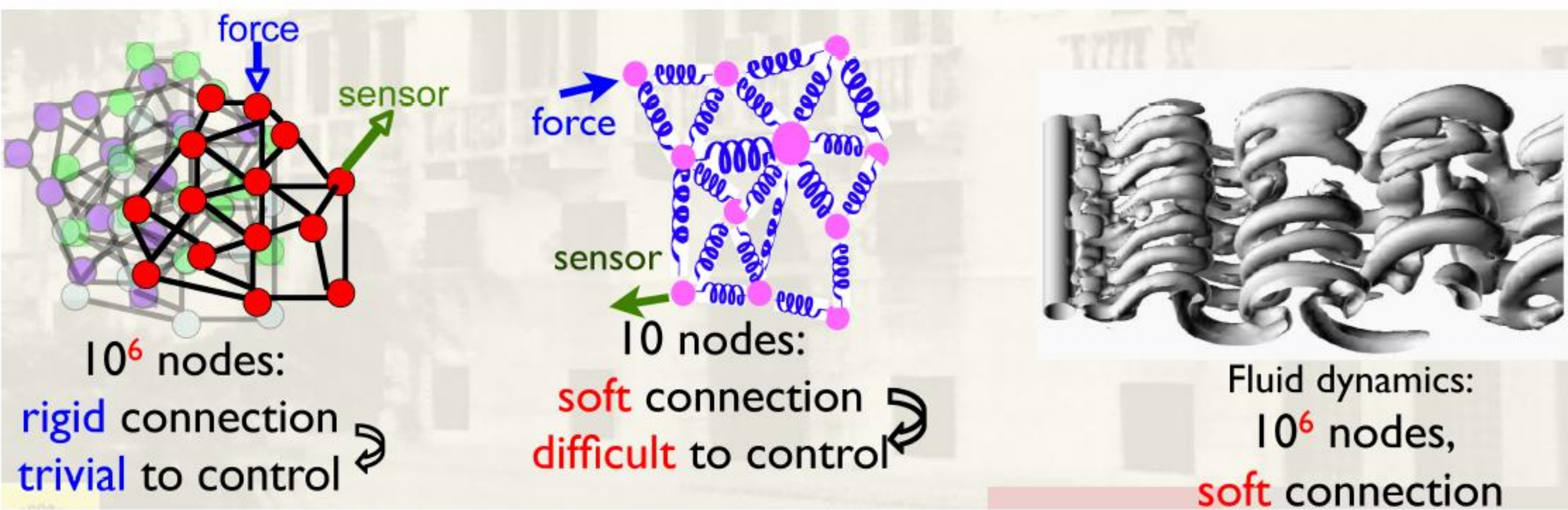
Fig. : *From Delville (2006)*

Feedback flow control. Why is it hard?

From G. Tadmor (Northeastern University, U.S.A.)

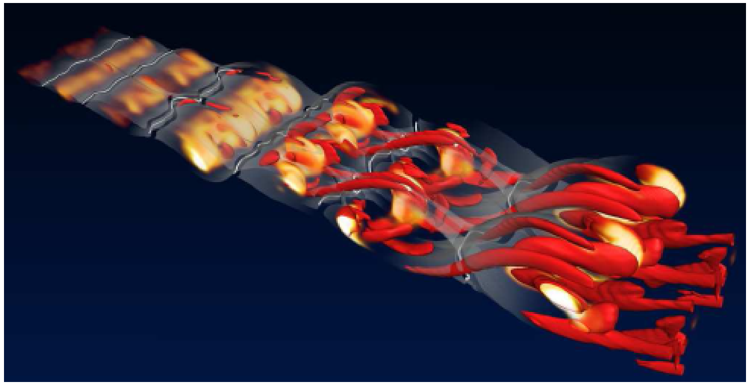
- "Fluid is not rigid" (a control's guy way to say it)
- The difficulty to control depends on how "flexible" it really is

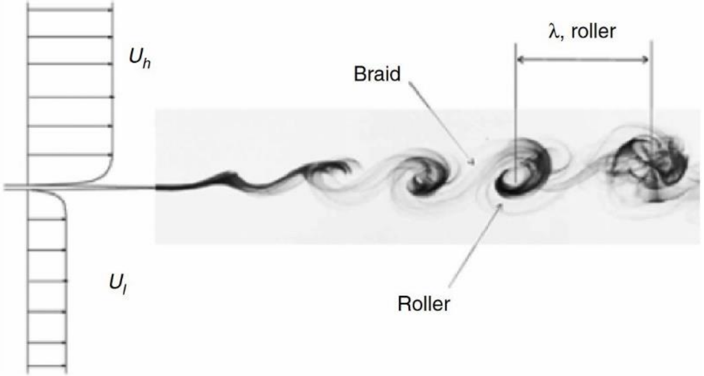
Intuitively:



Feedback flow control.

1. What makes it hard?
   - More "flexible" interconnections imply
     - more possible node configurations $\Longrightarrow$ harder to estimate the state
     - more possible frequencies $\Longrightarrow$ harder to estimate the phase
     - potential for chaotic / stochastic behavior / hysterisis
     - difficult to force the flow to behave as desired
     - difficult to compute (even define) best actuation
2. What makes it feasible?
   - Flow physics ensures that "some rigidity is maintained" at varying levels in different configurations
     - few persistent dominant coherent structures
     - distinct frequencies
     - well defined inertial manifold
     - sensitivity to actuation along a manifold
     - insensitivity to disturbance pushing away from manifold

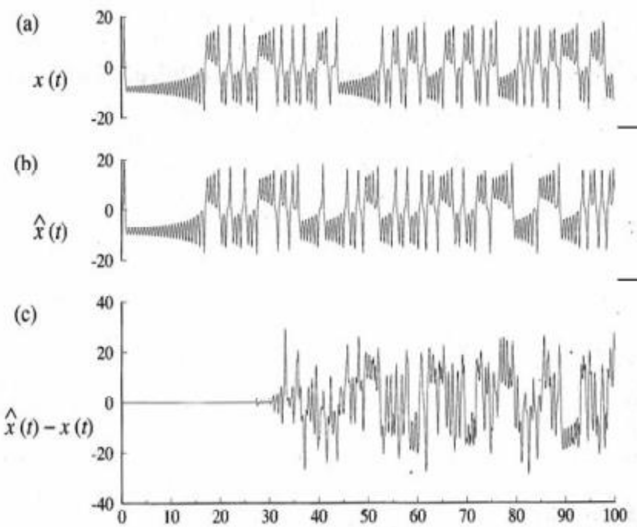




- Lorenz model (1963)

$$\dot{x} = \sigma(y - x) \; ; \quad \dot{y} = \rho . x - y - x . z \; ; \quad \dot{z} = -\beta . z + x . y$$

  - For $\sigma=10$, $\beta=8/3$ and $\rho=28$ ➔ chaotic behaviour is observed.



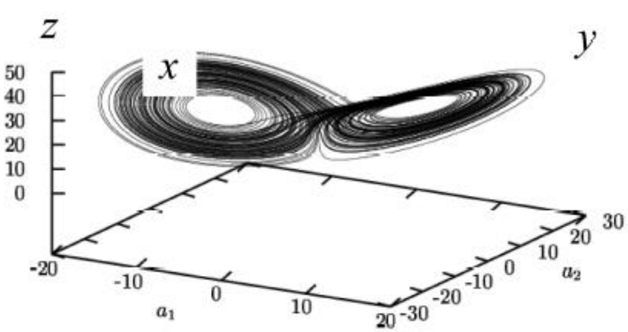
$x(0) = 0.1$
$y(0) = 0.1$
$z(0) = 0.1$

$\hat{x}(0) = 0.100001$
$\hat{y}(0) = 0.1$
$\hat{z}(0) = 0.1$

Difference

(POPE p.36)

Time histories from the Lorenz equations : (a) $x(t)$ from the initial condition Eq. (3.2); (b) $\hat{x}(t)$ from the slightly different initial condition and (c) the difference $\hat{x}(t) - x(t)$.

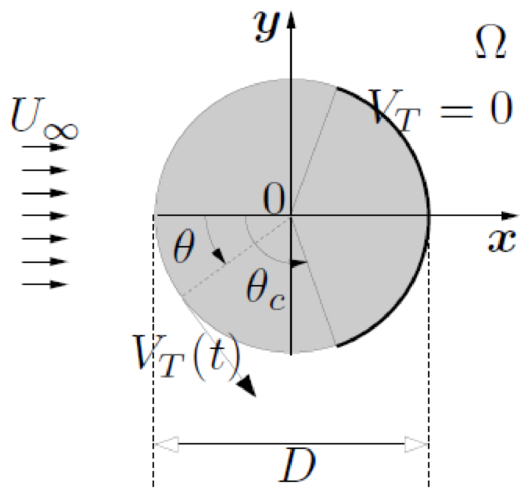


Strange Attractor

Flow control and optimization methods

- Two dimensional incompressible flow around a circular cylinder at $Re = 200$
- Cylinder oscillation with a tangential sinusoidal velocity $\gamma(t)$

$$\gamma(t) = \frac{V_T}{U_\infty} = A\sin(2\pi St_f t)$$



▷ Optimization framework:

Find the control parameters $c = (\theta_c,\ A,\ St_f)^T$ such that the mean drag coefficient is minimized

Flow control

Reverse Von Kármán street [Phys. Fluids 18 (2006)]

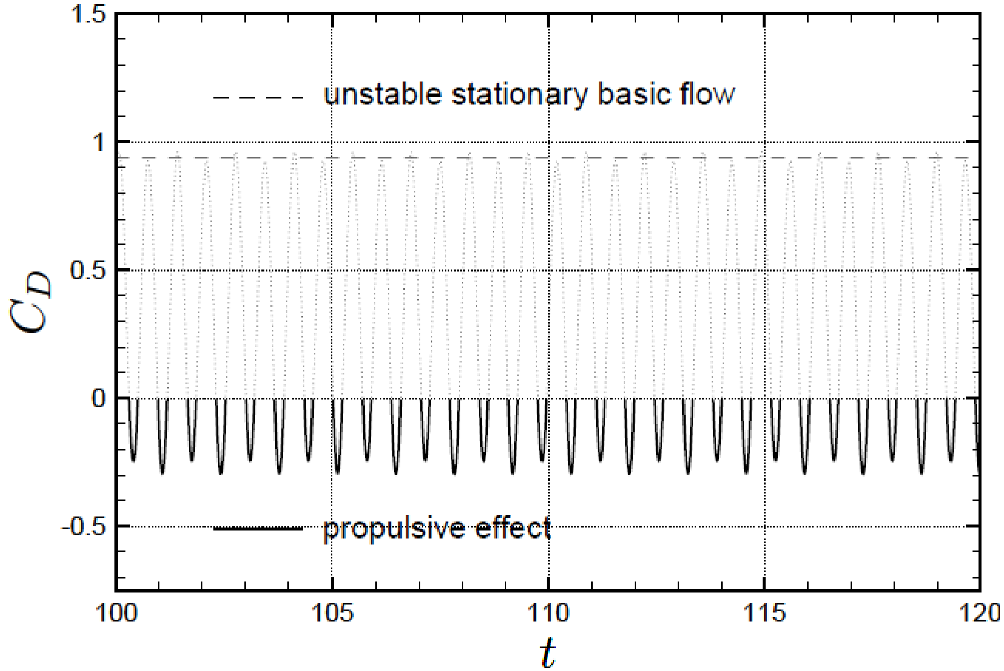


Fig. : *Drag evolution with* $\theta_c = 120$ *and* $(A, St_f) = (4.3, 0.74)$.

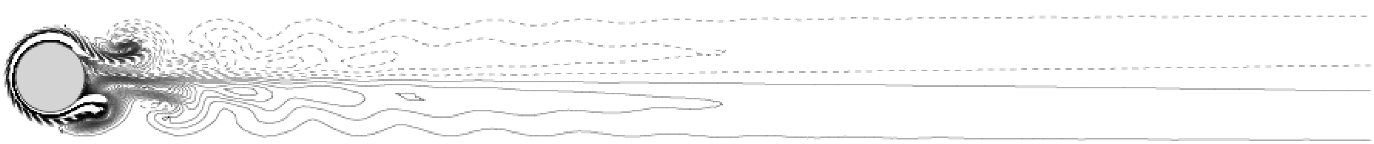
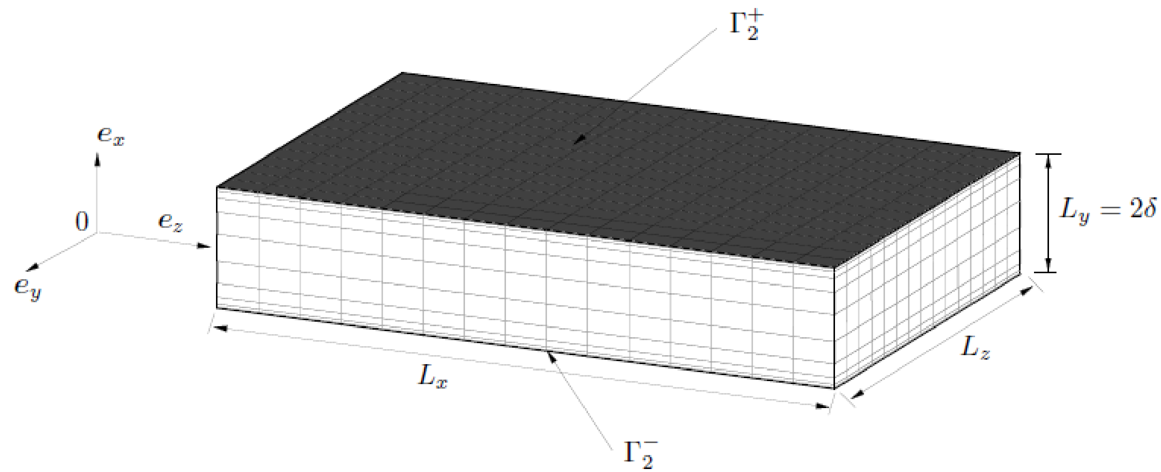


Fig. : *Whole wake.*

Movie

Optimal control

- Viscous, incompressible and Newtonian fluid
- Periodicity in $\boldsymbol{e}_x$ and $\boldsymbol{e}_z$
- Geometry $[L_x, 2\delta, L_z]$
- $Re_\tau = \dfrac{u_\tau\,\delta}{\nu} \in \{100; 180\}$
  with
  $$u_\tau = \sqrt{\frac{\langle \tau_w \rangle}{\rho}}.$$



- Boundary control
  $$\boldsymbol{u} = -\Phi \boldsymbol{n} \quad \text{on} \quad \Gamma_2^+ \quad \text{and} \quad \Gamma_2^-$$
  - $\Phi = 0$: uncontrolled flow
  - $\Phi > 0$: blowing
  - $\Phi < 0$: suction

Determine $c = (\Phi)$ such that the mean drag coefficient is minimized

with

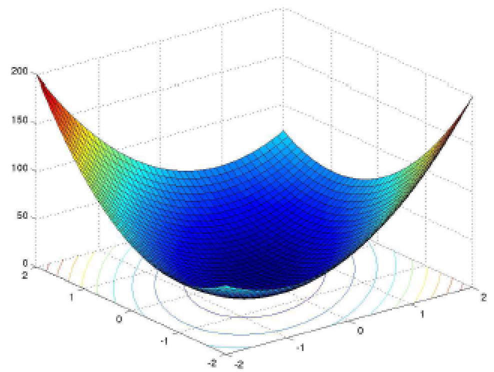
$$\Phi = \Phi(\boldsymbol{x}, \boldsymbol{y}, t).$$

Optimization methods

Minimize $\mathcal{J}$ with $\boldsymbol{c}$ control parameters

1. Deterministic algorithms of optimization
   - with evaluation of the gradient $\frac{d\,\mathcal{J}}{d\,\boldsymbol{c}}$: descent algorithms (Newton or quasi-Newton methods, ...)
     - Model-free approaches: extremum and slope seeking
     - Model-based approaches: optimal control
   - without evaluation of the gradient: simplex algorithm (no proof of convergence in dimension > 2)

2. Stochastic algorithms of optimization: Genetic Algorithms, Particle Swarm Optimization, Genetic Programming (Machine Learning Control). . .



Comparison of optimization algorithms

Descent algorithms		Genetic algorithms	
Positive points	Negative points	Positive points	Negative points
• Speed • Accuracy • Cost independent on the number of variables	• Local minimum • Gradient difficult to determine • No multi-objective version	• Global minimum • No regularity condition on $\mathcal{J}$ • Multi-objective • Parallel version	• Computing time • Cost dependent on the number of variables • Adjustable parameters

Chapter 1

Optimal control

Problem of interest:

$$\begin{aligned} &\text{Minimize } \mathcal{J}(q,g) \qquad \text{with} \quad q \in \mathbb{R}^N \quad \text{and} \quad g \in \mathbb{R}^K && \text{Cost function} \\ &\text{Subject to } F(q,g) = 0 && \text{Constraint} \end{aligned}$$

Vocabulary:

- State variable: q
- Control variable: g
- Cost function: $\mathcal{J} = \mathcal{J}(q,g) = \mathcal{J}(q(g),g)$
- State equation: $F = F(q,g) = F(q(g),g)$

1.1 Successive steps

1. $q \in \mathbb{R}\,; g \in \mathbb{R}\,; F \in \mathbb{R}$
   Use the constraint equation ($F = 0$) to eliminate a variable.

2. $q \in \mathbb{R}\,; g \in \mathbb{R}\,; F \in \mathbb{R}$
   Determine the **total derivative** $\dfrac{D\mathcal{J}}{Dg}$ through the estimation of the **sensitivity** $\dfrac{dq}{dg}$.
   $\Longrightarrow$ Differentiate $F = 0$

3. $\mathbf{q} \in \mathbb{R}^N\,; \mathbf{g} \in \mathbb{R}^K\,; \mathbf{F} \in \mathbb{R}^N$
   Determine the vectorial **total derivative** $\dfrac{D\mathcal{J}}{D\mathbf{g}}$ through the estimation of the **sensitivity**.

4. Back to $q \in \mathbb{R}\,; g \in \mathbb{R}\,; F \in \mathbb{R}$
   Intuition of the variational formulation.

5. Back to $\mathbf{q} \in \mathbb{R}^N\,; \mathbf{g} \in \mathbb{R}^K\,; \mathbf{F} \in \mathbb{R}^N$
   Generalization of the variational formulation. Inner product in $\mathbb{R}^N$.

6. Back to $\mathbf{q} \in \mathbb{R}^N\,; \mathbf{g} \in \mathbb{R}^K\,; \mathbf{F} \in \mathbb{R}^N$
   Generalization to function gradients.

7. Gradient method based on the adjoint equation.

8. Example: Optimal amplification of forcing with a steady linear system.

9. Introduce time dependent problems.
   Optimal energy growth.

10. Example: Final target state with time-dependent forcing.

11. Generalization to space-dependent functions. Introduction of **adjoint** operators.

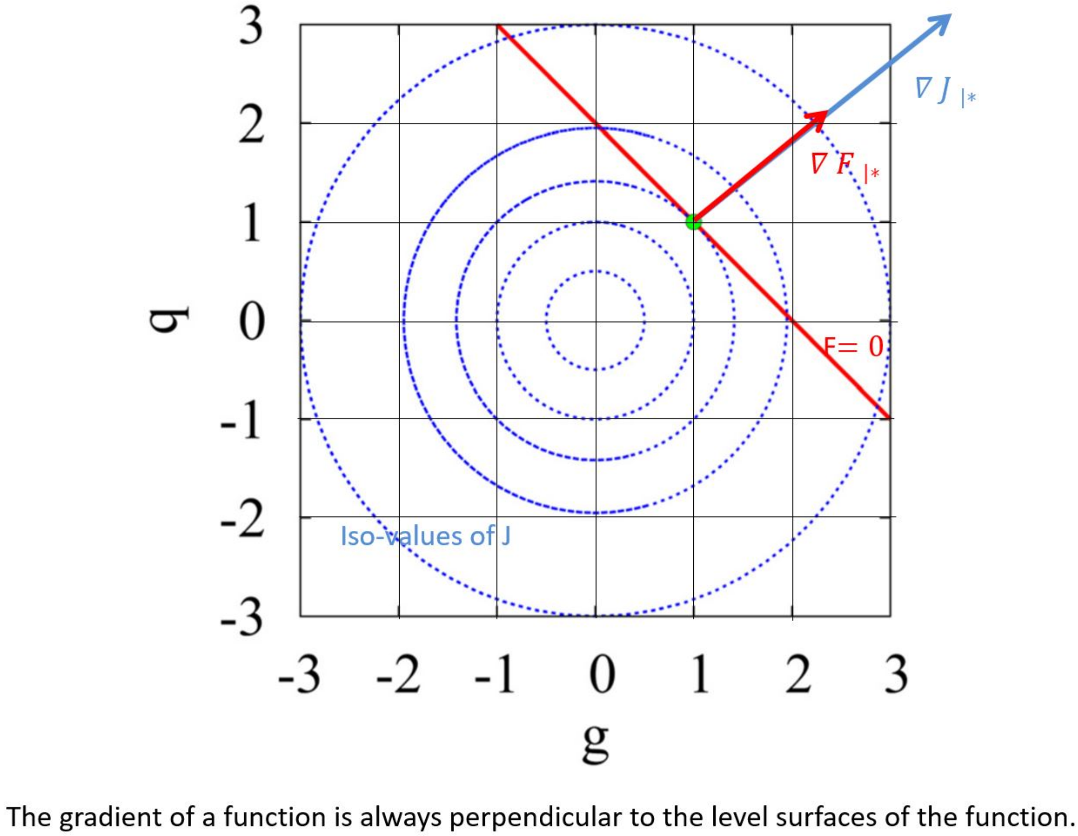


Figure 1.1: Iso-values of $\mathcal{J}(q, g) = q^2 + g^2$.

1.2 Step #1: Intuitive method: Eliminate a variable from $F = 0$

Introductory example: From Cossu (Applied Mechanics Review, 2014):

$$\text{Minimize } \mathcal{J}(q, g) = q^2 + g^2 \quad \text{with} \quad q \in \mathbb{R}\,;\, g \in \mathbb{R}$$
$$\text{Subject to } F(q, g) = q + g - 2 = 0$$

Unconstrained minimum: $(0, 0)$
Constrained minimum: $(1, 1)$ (graphic reading)

$F = 0 \Longrightarrow F = q + g - 2 = 0 \Longrightarrow q = 2 - g \Longrightarrow \mathcal{J}(q, g) = (2 - g)^2 + g^2 =$

$2g^2 - 4g + 4$

Necessary condition of extremum: Total derivative equal to 0

$$\frac{D\mathcal{J}}{Dg} = 4(g-1) = 0 \text{ for } g^* = 1 \Longrightarrow q^* = 1 \Longrightarrow \mathcal{J}^* = 2$$

$$\frac{D^2\mathcal{J}}{Dg^2} = 4 > 0 \Longrightarrow (1,1) \text{ is a minimum}$$

What can be done if we cannot resolve explicitly $F = 0$ *i.e.* determine $q(g)$?

1.3 Step #2: Determination total derivative through the sensitivity $\frac{dq}{dg}$

$$\mathcal{J}(q(g),g) \overset{\text{Chain rule}}{\Longrightarrow} \frac{D\mathcal{J}}{Dg} = \underbrace{\frac{\partial \mathcal{J}}{\partial q}}_{2q} \frac{dq}{dg} + \underbrace{\frac{\partial \mathcal{J}}{\partial g}}_{2g} = 0$$

How to determine $\frac{dq}{dg}$? We differentiate $F = 0 \Longrightarrow$

$$dF = \frac{\partial F}{\partial q} dq + \frac{\partial F}{\partial g} dg$$
$$\frac{dq}{dg} = -\left(\frac{\partial F}{\partial q}\right)^{-1} \frac{\partial F}{\partial g}$$

Here, $\frac{\partial F}{\partial q} = 1$ and $\frac{\partial F}{\partial g} = 1 \Longrightarrow \frac{dq}{dg} = -1 \Longrightarrow \frac{D\mathcal{J}}{Dg} = 2(g-q) \Longrightarrow \frac{D\mathcal{J}}{Dg} = 0$ for $g = q$.

Since $F(q,g) = q + g - 2 = 0 \Longrightarrow 2q = 2 \Longrightarrow q^* = 1 \quad ; \quad g^* = 1 \quad ; \quad \mathcal{J}^* = 2.$

1.4 Step #3: Determination vectorial total derivative $\frac{D\mathcal{J}}{D\mathbf{g}}$ through the sensitivities

We suppose $\mathbf{q} \in \mathbb{R}^N; \mathbf{F} \in \mathbb{R}^N; \mathbf{g} \in \mathbb{R}^K$ *i.e.* $\mathbf{g} = \sum_{k=1}^{K} g_k \mathbf{e}_k$ with $\mathbf{e}_k$ vectors of an orthonormal basis.

Total derivative:

$$\frac{D\mathcal{J}}{D\mathbf{g}} = \begin{pmatrix} \frac{D\mathcal{J}}{Dg_1} \\ \vdots \\ \frac{D\mathcal{J}}{Dg_K} \end{pmatrix} \in \mathbb{R}^K$$

i) Naive approximation of $\dfrac{D\mathcal{J}}{Dg_k}, k = 1, \cdots, K$ by finite differences

$$\frac{D\mathcal{J}}{Dg_k} \approx \frac{\mathcal{J}\left(\mathbf{q}\left(\mathbf{g} + \Delta g_k \mathbf{e}_k\right), \mathbf{g} + \Delta g_k \mathbf{e}_k\right) - \mathcal{J}(\mathbf{q}(\mathbf{g}), \mathbf{g})}{\Delta g_k}$$

with Δg_k small increment.

Drawbacks:

- $K + 1$ resolution of $\mathbf{F} = \mathbf{0}$ are needed ; costly if solving $\mathbf{F} = \mathbf{0}$ is costly.
- prone to numerical inaccuracy: how to choose Δg_k?

ii) Generalization of the previous approach: sensitivity equations

$$\begin{aligned} \frac{D\mathcal{J}}{Dg_k} &= \sum_{i=1}^{N} \frac{\partial \mathcal{J}}{\partial q_i} \frac{dq_i}{dg_k} + \frac{\partial \mathcal{J}}{\partial g_k} \\ &= \frac{\partial \mathcal{J}}{\partial \mathbf{q}} \cdot \frac{d\mathbf{q}}{dg_k} + \frac{\partial \mathcal{J}}{\partial g_k} \end{aligned}$$

with

$$\frac{\partial \mathcal{J}}{\partial \mathbf{q}} = \begin{pmatrix} \frac{\partial \mathcal{J}}{\partial q_1} \\ \vdots \\ \frac{\partial \mathcal{J}}{\partial q_N} \end{pmatrix} \in \mathbb{R}^N \quad \text{and} \quad \frac{d\mathbf{q}}{dg_k} = \begin{pmatrix} \frac{dq_1}{dg_k} \\ \vdots \\ \frac{dq_N}{dg_k} \end{pmatrix} \in \mathbb{R}^N$$

We differentiate $\mathbf{F}(\mathbf{q},\mathbf{g}) = 0 \Longrightarrow \dfrac{D\mathbf{F}}{Dg_k} = 0,\ \forall k = 1, \cdots, K \quad \Longrightarrow$

$$\frac{\partial \mathbf{F}}{\partial \mathbf{q}}\frac{d\mathbf{q}}{dg_k}+\frac{\partial \mathbf{F}}{\partial g_k} = 0 \quad \text{sensitivity eqs.} \quad \text{with} \quad \frac{\partial \mathbf{F}}{\partial \mathbf{q}} = \begin{pmatrix} \frac{\partial F_1}{\partial q_1}, \cdots, \frac{\partial F_1}{\partial q_N} \\ \vdots \\ \frac{\partial F_N}{\partial q_1}, \cdots, \frac{\partial F_N}{\partial q_N} \end{pmatrix} \text{Jacobian matrix}$$

$$\boxed{\frac{d\mathbf{q}}{dg_k} = -\left(\frac{\partial \mathbf{F}}{\partial \mathbf{q}}\right)^{-1}\frac{\partial \mathbf{F}}{\partial g_k}}$$

This is the **sensitivity equations** method. $\dfrac{d\mathbf{q}}{dg_k}$ is obtained as a solution of a linear system of equations of size $N \times N$ (costly).

1.5 Step #4: Intuition of the variational formulation

At the minimum of F, $F = 0$ is tangent to the iso-value of the minimum. We have:

$$\nabla F|_* \propto \nabla \mathcal{J}|_* \quad \text{and} \quad F = 0 \Longrightarrow$$

$$\exists a \quad \text{s.t.} \quad \begin{pmatrix} \dfrac{\partial \mathcal{J}}{\partial q} \\ \\ \dfrac{\partial \mathcal{J}}{\partial g} \end{pmatrix} = a \begin{pmatrix} \dfrac{\partial F}{\partial q} \\ \\ \dfrac{\partial F}{\partial g} \end{pmatrix} \Longrightarrow \quad \text{Optimality system}$$

$$\begin{aligned} \frac{\partial \mathcal{J}}{\partial q} - a\frac{\partial F}{\partial q} &= 0 \quad \text{Adjoint equation} \\ \frac{\partial \mathcal{J}}{\partial g} - a\frac{\partial F}{\partial g} &= 0 \quad \text{Optimlity condition} \\ F &= 0 \quad \text{Constraint} \end{aligned}$$

Lagrange noticed that this system of equations corresponded to the optimal system that could be written for an augmented Lagrangian defined by:

$$\mathcal{L}(q, g, a) = \mathcal{J} - aF \quad \text{with}$$

considering the variables q, g, a as independent.

Vocabulary:

- Lagrangian or augmented cost function: $\mathcal{L}$
- Co-state or Lagrange multiplier: a

Ex : Cossu (AMR, 2014). In $\mathbb{R}$

$$\frac{\partial \mathcal{J}}{\partial q} = 2q \quad ; \quad \frac{\partial \mathcal{J}}{\partial g} = 2g \quad ; \quad \frac{\partial F}{\partial q} = 1q \quad ; \quad \frac{\partial F}{\partial g} = 1q \quad ;$$

$$\begin{aligned} \text{Optimality system :} \quad & 2q - a = 0 \quad ; \quad 2g - a = 0 \quad ; \quad q + g - 2 = 0 \\ \text{Eliminate } a \text{ :} \quad & a = 2q = 2g \Longrightarrow q = g \Longrightarrow q = g = 1 \\ \text{Solution :} \quad & (q, g, a)_* = (1, 1, 2) \quad ; \quad \mathcal{J}_* = 2 \end{aligned}$$

1.6 Step #5: Generalization of the variational formulation. Inner product in $\mathbb{R}^N$

We introduce a Lagrange multiplier a_j for each component F_j ($j = 1, \cdots, N$). we have:

$$\mathcal{L}(\mathbf{q}, \mathbf{g}, \mathbf{a}) = \mathcal{J}(\mathbf{q}, \mathbf{g}) - \sum_{j=1}^{N} a_j F_j(\mathbf{q}, \mathbf{g}) = \mathcal{J}(\mathbf{q}, \mathbf{g}) - \mathbf{a} \cdot \mathbf{F}(\mathbf{q}, \mathbf{g})$$

Optimality is obtained by considering that the variables $\mathbf{q}, \mathbf{g}$ and $\mathbf{a}$ are independent, *i.e.*

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} = 0 \quad ; \quad \frac{\partial \mathcal{L}}{\partial \mathbf{g}} = 0 \quad ; \quad \frac{\partial \mathcal{L}}{\partial \mathbf{a}} = 0$$

We obtain component by component: $2N + K$ equations

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\partial \mathcal{J}}{\partial q_i} - \sum_j a_j \frac{F_j}{\partial q_i} = 0 \qquad i = 1, \cdots, N$$

$$\frac{\partial \mathcal{L}}{\partial g_k} = \frac{\partial \mathcal{J}}{\partial g_k} - \sum_j a_j \frac{F_j}{\partial g_k} = 0 \qquad k = 1, \cdots, K$$

$$\frac{\partial \mathcal{L}}{\partial a_i} = -F_i = 0 \qquad i = 1, \cdots, N$$

Rk: $\sum_j a_j \frac{F_j}{\partial q_i} = \sum_i a_i \frac{F_i}{\partial q_j} \Longrightarrow$ Optimality system in vectorial notation:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} = 0 \quad \Longrightarrow \quad \left(\frac{\partial \mathbf{F}}{\partial \mathbf{q}}\right)^T \mathbf{a} = \frac{\partial \mathcal{J}}{\partial \mathbf{q}} \qquad \text{Adjoint equation}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{g}} = 0 \quad \Longrightarrow \quad \left(\frac{\partial \mathbf{F}}{\partial \mathbf{g}}\right)^T \mathbf{a} = \frac{\partial \mathcal{J}}{\partial \mathbf{g}} \qquad \text{Optimality condition}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{a}} = 0 \quad \Longrightarrow \quad \mathbf{F} = \mathbf{0} \qquad \text{State equation}$$

1.7 Step #6: Generalization to function gradients ($\mathbb{R}^N$)

We consider the first order variation $\delta\mathcal{L}$ introduced by small variations $\delta\mathbf{q}$, $\delta\mathbf{g}$ and $\delta\mathbf{a}$. At the optimum, we have:

$$\frac{\partial\mathcal{L}}{\partial\mathbf{q}} = 0 \quad ; \quad \frac{\partial\mathcal{L}}{\partial\mathbf{g}} = 0 \quad ; \quad \frac{\partial\mathcal{L}}{\partial\mathbf{a}} = 0$$

By definition, the variation $\delta\mathcal{L}$ induced by a small variation $\delta\mathbf{a} = \epsilon\tilde{\mathbf{a}}$ with ϵ a small parameter, is given by the directional derivative:

$$\frac{\partial\mathcal{L}}{\partial\mathbf{a}}\tilde{\mathbf{a}} \triangleq \lim_{\epsilon\longrightarrow 0} \frac{\mathcal{L}(\mathbf{q},\mathbf{g},\mathbf{a}+\epsilon\tilde{\mathbf{a}}) - \mathcal{L}(\mathbf{q},\mathbf{g},\mathbf{a})}{\epsilon}$$

Reminder: $\mathcal{L}(\mathbf{q},\mathbf{g},\mathbf{a}) = \mathcal{J}(\mathbf{q},\mathbf{g}) - \sum_{j=1}^{N} a_j F_j(\mathbf{q},\mathbf{g}) = \mathcal{J}(\mathbf{q},\mathbf{g}) - \mathbf{a}\cdot\mathbf{F}(\mathbf{q},\mathbf{g})$

▷ Variation with respect to the co-state $\mathbf{a}$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\mathbf{a}}\tilde{\mathbf{a}} &= \lim_{\epsilon\longrightarrow 0} -\frac{1}{\epsilon}\left[(\mathbf{a}+\epsilon\tilde{\mathbf{a}})\cdot\mathbf{F} - \mathbf{a}\cdot\mathbf{F}\right] \\ &= -\tilde{\mathbf{a}}\cdot\mathbf{F} = 0 \quad \forall\tilde{\mathbf{a}} \quad \Longrightarrow \mathbf{F} = \mathbf{0} \quad \text{State equation}\end{aligned}$$

▷ Variation with respect to the state $\mathbf{q}$

$$\frac{\partial\mathcal{L}}{\partial\mathbf{q}}\tilde{\mathbf{q}} = \lim_{\epsilon\longrightarrow 0} \frac{\mathcal{L}(\mathbf{q}+\epsilon\tilde{\mathbf{q}},\mathbf{g},\mathbf{a}) - \mathcal{L}(\mathbf{q},\mathbf{g},\mathbf{a})}{\epsilon}$$

$$\begin{aligned}\mathcal{L}(\mathbf{q}+\epsilon\tilde{\mathbf{q}},\mathbf{g},\mathbf{a}) &= \mathcal{J}(\mathbf{q}+\epsilon\tilde{\mathbf{q}},\mathbf{g}) - \mathbf{a}\cdot\mathbf{F}(\mathbf{q}+\epsilon\tilde{\mathbf{q}},\mathbf{g}) \\ \mathcal{L}(\mathbf{q},\mathbf{g},\mathbf{a}) &= \mathcal{J}(\mathbf{q},\mathbf{g}) - \mathbf{a}\cdot\mathbf{F}(\mathbf{q},\mathbf{g})\end{aligned}$$

We perform Taylor developments to order 2.

$$\begin{aligned}\mathcal{J}(\mathbf{q}+\epsilon\tilde{\mathbf{q}},\mathbf{g}) &= \mathcal{J}(\mathbf{q},\mathbf{g}) + \epsilon\frac{\partial\mathcal{J}}{\partial\mathbf{q}}\cdot\tilde{\mathbf{q}} + \mathcal{O}(\epsilon^2) \\ \mathbf{F}(\mathbf{q}+\epsilon\tilde{\mathbf{q}},\mathbf{g}) &= \mathbf{F}(\mathbf{q},\mathbf{g}) + \epsilon\frac{\partial\mathbf{F}}{\partial\mathbf{q}}\tilde{\mathbf{q}} + \mathcal{O}(\epsilon^2)\end{aligned}$$

$$\Longrightarrow \frac{\partial\mathcal{L}}{\partial\mathbf{q}}\tilde{\mathbf{q}} = \frac{\partial\mathcal{J}}{\partial\mathbf{q}}\cdot\tilde{\mathbf{q}} - \mathbf{a}\cdot\left(\frac{\partial\mathbf{F}}{\partial\mathbf{q}}\right)\tilde{\mathbf{q}}$$

Rk: $\mathbf{u} \cdot A\mathbf{v} = A^T\mathbf{u} \cdot \mathbf{v} \quad \Longrightarrow$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}}\tilde{\mathbf{q}} = \frac{\partial \mathcal{J}}{\partial \mathbf{q}} \cdot \tilde{\mathbf{q}} - \left(\frac{\partial \mathbf{F}}{\partial \mathbf{q}}\right)^T \mathbf{a} \cdot \tilde{\mathbf{q}}$$
$$= \left[\frac{\partial \mathcal{J}}{\partial \mathbf{q}} - \left(\frac{\partial \mathbf{F}}{\partial \mathbf{q}}\right)^T \mathbf{a}\right] \cdot \tilde{\mathbf{q}} = 0 \quad \forall \tilde{\mathbf{q}} \quad \Longrightarrow \frac{\partial \mathcal{J}}{\partial \mathbf{q}} - \left(\frac{\partial \mathbf{F}}{\partial \mathbf{q}}\right)^T \mathbf{a} = \mathbf{0} \quad \text{Adjoint equation}$$

▷ Variation with respect to the control $\mathbf{g}$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{g}}\tilde{\mathbf{g}} = \lim_{\epsilon \longrightarrow 0} \frac{\mathcal{L}(\mathbf{q}, \mathbf{g} + \epsilon\tilde{\mathbf{g}}, \mathbf{a}) - \mathcal{L}(\mathbf{q}, \mathbf{g}, \mathbf{a})}{\epsilon}$$
$$\frac{\partial \mathcal{J}}{\partial \mathbf{g}} \cdot \tilde{\mathbf{g}} - \mathbf{a} \cdot \left(\frac{\partial \mathbf{F}}{\partial \mathbf{g}}\right) \tilde{\mathbf{g}}$$
$$= \left[\frac{\partial \mathcal{J}}{\partial \mathbf{g}} - \left(\frac{\partial \mathbf{F}}{\partial \mathbf{g}}\right)^T \mathbf{a}\right] \cdot \tilde{\mathbf{g}} = 0 \quad \forall \tilde{\mathbf{g}} \quad \Longrightarrow \frac{\partial \mathcal{J}}{\partial \mathbf{g}} - \left(\frac{\partial \mathbf{F}}{\partial \mathbf{g}}\right)^T \mathbf{a} = \mathbf{0} \quad \text{Optimality condition}$$

In summary, we obtain the optimality system composed of:

$$\mathbf{F} = \mathbf{0} \quad \text{State equation}$$
$$\frac{\partial \mathcal{J}}{\partial \mathbf{q}} - \left(\frac{\partial \mathbf{F}}{\partial \mathbf{q}}\right)^T \mathbf{a} = \mathbf{0} \quad \text{Adjoint equation}$$
$$\frac{\partial \mathcal{J}}{\partial \mathbf{g}} - \left(\frac{\partial \mathbf{F}}{\partial \mathbf{g}}\right)^T \mathbf{a} = \mathbf{0} \quad \text{Optimality condition}$$

This is a system of coupled equations that are solved by *one shot method* or *iterative method*.

1.8 Step #7: Gradient method based on the adjoint equation

Objective: Show that the use of adjoint equations decreases the number of operations needed to calculate the total derivative.

We showed previously that:

$$\begin{aligned}\frac{D\mathcal{J}}{Dg_k} &= \frac{\partial \mathcal{J}}{\partial \mathbf{q}} \cdot \frac{d\mathbf{q}}{dg_k} + \frac{\partial \mathcal{J}}{\partial g_k} \\ &= \sum_{i=1}^{N} \frac{\partial \mathcal{J}}{\partial q_i} \frac{dq_i}{dg_k} + \frac{\partial \mathcal{J}}{\partial g_k}\end{aligned}$$

We search to simplify this expression.

1. Introduction of the adjoint equation $\dfrac{\partial \mathcal{J}}{\partial q_i} = \sum_j a_j \dfrac{\partial F_j}{\partial q_i} \Longrightarrow$

$$\frac{D\mathcal{J}}{Dg_k} = \sum_i \sum_j a_j \frac{\partial F_j}{\partial q_i} \frac{dq_i}{dg_k} + \frac{\partial \mathcal{J}}{\partial g_k}$$

2. Let's differentiate $\mathbf{F}(\mathbf{q}, \mathbf{g}) = 0 \Longrightarrow \dfrac{D\mathbf{F}}{Dg_k} = 0,\ \forall k \Longrightarrow \dfrac{\partial \mathbf{F}}{\partial \mathbf{q}} \dfrac{d\mathbf{q}}{dg_k} + \dfrac{\partial \mathbf{F}}{\partial g_k} = 0$

   We deduce for the j-th component: $\displaystyle\sum_i \frac{\partial F_j}{\partial q_i} \frac{dq_i}{dg_k} = -\frac{\partial F_j}{\partial g_k}.$

   Finally, we obtain: $$\frac{D\mathcal{J}}{Dg_k} = -\sum_j a_j \frac{\partial F_j}{\partial g_k} + \frac{\partial \mathcal{J}}{\partial g_k}$$

   The vectorial expression is

$$\frac{D\mathcal{J}}{D\mathbf{g}} = -\left(\frac{\partial \mathbf{F}}{\partial \mathbf{g}}\right)^T \mathbf{a} + \frac{\partial \mathcal{J}}{\partial \mathbf{g}} \quad \text{with} \quad \left(\frac{\partial \mathbf{F}}{\partial \mathbf{q}}\right)^T \mathbf{a} = \frac{\partial \mathcal{J}}{\partial \mathbf{q}} \quad \text{Adjoint eq.}$$

   If we know $\dfrac{\partial \mathcal{J}}{\partial \mathbf{q}}$, $\dfrac{\partial \mathcal{J}}{\partial \mathbf{g}}$, $\dfrac{\partial \mathbf{F}}{\partial \mathbf{q}}$ and $\dfrac{\partial \mathbf{F}}{\partial \mathbf{g}}$, then the cost of determining the total derivative is given by the cost of solving the adjoint equation. This cost corresponds to the solution of one linear system of equations of size N. Then, this is much less costly to use the adjoint equations for determining the total derivative.

1.9 Step #8: Example: Optimal amplification of forcing with a steady linear system

$$\text{Consider} \qquad Lu + f = 0$$

We look for f (steady) that maximizes the energy amplification defined as

$$R = \frac{u \cdot u}{f \cdot f}$$

Rk 1: Maximizing R is equivalent to minimizing $1/R$.
Rk 2: Formally, we have:

$$u = -L^{-1} f \quad \Longrightarrow \quad R = \frac{L^{-1} f \cdot L^{-1} f}{f \cdot f} = \frac{\|L^{-1} f\|^2}{\|f\|^2}$$

$$\max_{f \neq 0} R = \max_{f \neq 0} \frac{\|L^{-1} f\|^2}{\|f\|^2} \triangleq \left\|L^{-1}\right\|^2$$

For solving the maximization problem over f, we use the **optimal control** approach for:

$$q \equiv u \quad ; \quad g \equiv f \quad \text{and} \quad K = N$$

$F(q, g) = Lq + g$ State equation

$\mathcal{J}(q, g) = \dfrac{g \cdot g}{q \cdot q} = \dfrac{1}{R}$ Cost function (minimization of $\mathcal{J}$ for maximization of R)

Optimality system:

$$F = 0 \quad \text{State equation}$$

$$\left(\frac{\partial F}{\partial q}\right)^T a = \frac{\partial \mathcal{J}}{\partial q} \quad \text{Adjoint equation}$$

$$\left(\frac{\partial F}{\partial g}\right)^T a = \frac{\partial \mathcal{J}}{\partial g} \quad \text{Optimality condition}$$

$$\frac{\partial F}{\partial q} = L \quad ; \quad \frac{\partial F}{\partial g} = I \quad ; \quad \frac{\partial \mathcal{J}}{\partial q} = -2q \frac{g \cdot g}{(q \cdot q)^2} \quad ; \quad \frac{\partial \mathcal{J}}{\partial q} = \frac{2g}{q \cdot q} \quad ;$$

$$\text{Adjoint equation} \Longrightarrow \qquad L^T a = -2q\frac{g \cdot g}{(q \cdot q)^2} \qquad \text{Dim. } N$$

$$\text{Optimality condition} \Longrightarrow \qquad a = \frac{2g}{q \cdot q} \Longrightarrow g = \frac{1}{2}a\,(q \cdot q) \qquad \text{Dim. } K$$

$$\text{State equation} \Longrightarrow \qquad Lq + g = 0 \qquad \text{Dim. } N$$

Since $K = N$, the size of the optimality system is $2N + K = 3N$.

Iterative resolution of the optimality system

1. Given the n-th guess for the optimal forcing $g^{(n)}$, compute $q^{(n)}$ by solving the state equation:
$$Lq^{(n)} = -g^{(n)}$$
2. Compute $\mathcal{J}$ and $\Delta\mathcal{J}^{(n)}$ the increment of $\mathcal{J}$ between two iterations.
3. Compute the adjoint state $a^{(n)}$ solving the adjoint equation.
4. Determine $g^{(n+1)}$ using the optimality condition and go to (1).

Ex: $L = \begin{bmatrix} -\dfrac{1}{\mathrm{Re}} & 0 \\ 1 & -\dfrac{3}{\mathrm{Re}} \end{bmatrix}$

Rk 1: We define as $L^* = L^T$, the adjoint matrix of L. L is called a **non normal** matrix, since $LL^* \neq L^*L$.
Rk 2: In an Hermitian space, a matrix is normal if and only if it is diagonalizable in an orthonormal basis.

Rk 3: $\lambda(L) = \{-\frac{1}{\mathrm{Re}}; -\frac{3}{\mathrm{Re}}\}$. At least, one eigenvalue is strictly negative, then it means the linearized system based on L is stable. There is a decay at long time evolution of the solution.

Scilab program:
The optimality system is solved iteratively.
We consider a guess solution for the optimal forcing (g).

1. Solve the State equation

$$Lq + g = 0 \quad \Longrightarrow \quad q = -L^{-1}g$$

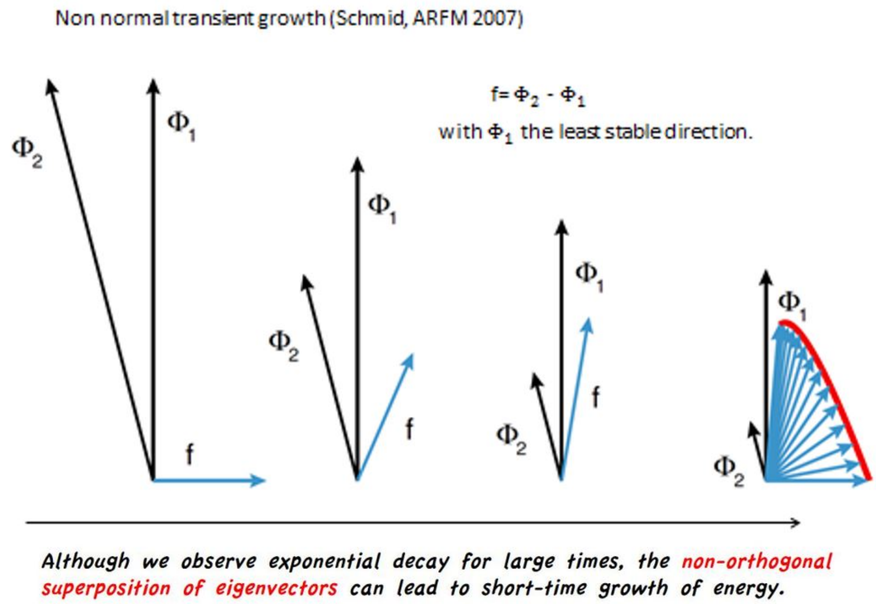


Figure 1.2: Non normal transient growth.

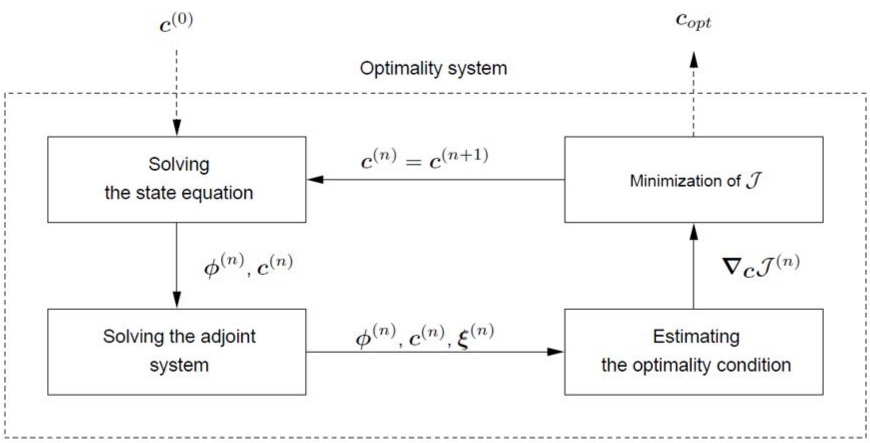


Figure 1.3: Iterative resolution of the optimality system.

2. Solve the Adjoint equation

$$L^T a = -2q \frac{g \cdot g}{(q \cdot q)^2} \quad \Longrightarrow \quad a$$

3. Solve the Optimality condition

$$g = \frac{1}{2} a \, (q \cdot q)$$

$$\mathcal{J}(q, g) = \frac{g \cdot g}{q \cdot q}$$

```
// Define system
// Reynolds number
Rey=40.0
// L matrix (L is non normal)
L=[-1.0/Rey 0; 1 -3.0/Rey]
// Exact solution using the norm function
R_exact=(norm(inv(-L)))^2
// Define tolerance and initialize iterations
// Tolerance for convergence
tol=10^(-8);
// Initialize control (random)
g=[rand(); rand()];
// Normalize g (optional)
//g=g/norm(g);
// Initialize J
J=10^23;
// Initialize dJrel = (J^{(n+1)}-J^{(n)})/J^{(n)} : relative variation
dJrel=10^23
// Initialize it : iteration number
it=0;
//
// Iteration loop
// While not converged
while (dJrel>tol)
it=it+1; Jold=J;
q=-inv(L)*g; // (solve state equation)
g2=g'*g; q2=q'*q;
J=g2/q2; // (objective function)
dJrel=abs((J-Jold)/J);
a=-2*(inv(L')*q)*g2/q2^2; // (solve adjoint equation)
g=a*q2/2.0; // (enforce optimality eq.)
// Normalize g (optional)
//g=g/norm(g);
end // (end of iteration loop)
// optimal amplification
R=1.0/J;
// print results
it, R // (final iteration and amplification)
g // (optimal forcing)
q // (optimal response)
// normalize g and q
g=g/norm(g)
q=q/norm(q)
```

CODE 1.1: Optimal amplification of forcing with a steady linear system.

$$\text{Optimal forcing: } \begin{pmatrix}1\\0\end{pmatrix} \quad ; \quad \text{Optimal response: } \begin{pmatrix}0\\1\end{pmatrix} \quad ; \quad R \longrightarrow 286221.23$$

1.10 Step #9: Example: Optimal energy growth

Introduction of time and of a non linear model. We consider two constraints:

$$
\begin{aligned}
F(q) = \frac{dq}{dt} - N(q) = 0 \qquad & \text{State equation} \\
F_0(q, g) = q(0) - g = 0 \qquad & \text{Initial condition}
\end{aligned}
$$

The control parameter is $g = q(0)$.

We search to maximize the temporal energy amplification:

$$G(T) = \frac{q(T) \cdot q(T)}{q(0) \cdot q(0)}$$

This is the ratio of the energy at time T with the energy at time 0. Maximizing G is equivalent to minimizing:

$$\mathcal{J}(q, g) = \frac{g \cdot g}{q(T) \cdot q(T)} = \left.\frac{g \cdot g}{q(t) \cdot q(t)}\right|_{t=T}$$

We have two constraints:

1. $\dfrac{dq}{dt} - N(q) = 0$
2. $q(0) - g = 0$

We follow the variational approach and modify the inner product since one of the constraint is depending on time. We introduce the following Lagrangian:

$$\mathcal{L}(q, g, a, b) = \mathcal{J}(q, g) - \int_0^T a(t) \underbrace{\left[\frac{dq}{dt} - N(q)\right]}_{F(q)} \mathrm{d}t - b \underbrace{[q(0) - g]}_{F_0(q,g)}$$

▷ **State equation**: variation with respect to the co-state a

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial a} \tilde{a} = \lim_{\epsilon \longrightarrow 0} \frac{\mathcal{L}(q, g, a + \epsilon \tilde{a}, b) - \mathcal{L}(q, g, a, b)}{\epsilon} = 0 \qquad & \forall \tilde{a} \\
\Longrightarrow \int_0^T \tilde{a}(t) \left[\frac{dq}{dt} - N(q)\right] \mathrm{d}t = 0 \qquad & \forall \tilde{a} \\
\Longrightarrow \frac{dq}{dt} = N(q) &
\end{aligned}
$$

▷ **State equation**: variation with respect to the co-state b

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial b}\tilde{b} &= \lim_{\epsilon \longrightarrow 0} \frac{\mathcal{L}(q,g,a,b+\epsilon\tilde{b}) - \mathcal{L}(q,g,a,b)}{\epsilon} = 0 \qquad \forall \tilde{b} \\ &\Longrightarrow \tilde{b}\,[q(0)-g] = 0 \qquad \forall \tilde{b} \\ &\Longrightarrow q(0) = g\end{aligned}$$

▷ **Adjoint equation**: variation with respect to the state q

$$\frac{\partial \mathcal{L}}{\partial q}\tilde{q} = \lim_{\epsilon \longrightarrow 0} \frac{\mathcal{L}(q+\epsilon\tilde{q},g,a,b) - \mathcal{L}(q,g,a,b)}{\epsilon} = 0 \qquad \forall \tilde{q}$$

$$\begin{aligned}\mathcal{L}(q+\epsilon\tilde{q},g,a,b) - \mathcal{L}(q,g,a,b) &= \mathcal{J}(q+\epsilon\tilde{q},g) - \mathcal{J}(q,g) \\ &\quad - \int_0^T a(t)\ [F(q+\epsilon\tilde{q}) - F(q)]\,\mathrm{d}t \\ &\quad - b\,[F_0(q+\epsilon\tilde{q},g) - F_0(q,g)]\end{aligned}$$

We perform Taylor developments to order 2, for $\mathcal{J}$, F and F_0.

$$\mathcal{J}(q+\epsilon\tilde{q},g) = \mathcal{J}(q,g) + \epsilon\frac{\partial \mathcal{J}}{\partial q}\cdot\tilde{q} + \mathcal{O}(\epsilon^2)$$

$$\frac{\partial \mathcal{L}}{\partial q}\tilde{q} = \frac{\partial \mathcal{J}}{\partial q(t)}\tilde{q}(t)\Big|_{t=T} - \int_0^T a(t)\left(\frac{\partial F}{\partial q}\right)\tilde{q}\,\mathrm{d}t - b\left(\frac{\partial F_0}{\partial q}\right)\tilde{q} = 0 \qquad \forall \tilde{q}$$

From the definition of F and F_0, we deduce:

$$\frac{\partial F}{\partial q} = \frac{dI}{dt} - \frac{\partial N}{\partial q} \quad \text{and} \quad \frac{\partial F_0}{\partial q} = I(0)$$

where I is the identity operator (for instance, $Iq = q$). We deduce:

$$\frac{\partial \mathcal{L}}{\partial q}\tilde{q} = \frac{\partial \mathcal{J}}{\partial q(T)}\tilde{q}(T) - \int_0^T a(t)\left(\frac{d\tilde{q}}{dt} - \frac{\partial N}{\partial q}\tilde{q}\right)\mathrm{d}t - b\,\tilde{q}(0) = 0 \qquad \forall \tilde{q} \qquad (1.1)$$

We now transform this expression with the goal to exploit the fact that this equations is verified $\forall \tilde{q}$. For that, we try to put $\tilde{q}$ in factor.

1. Since $\mathbf{u} \cdot A\mathbf{v} = A^T \mathbf{u} \cdot \mathbf{v}$, we can write:

$$\int_0^T a(t)\,\frac{\partial N}{\partial q}\tilde{q}\,\mathrm{d}t = \int_0^T \left(\frac{\partial N}{\partial q}\right)^T a \cdot \tilde{q}\,\mathrm{d}t$$

2. We integrate by parts:

$$\int_0^T a(t)\,\frac{d\tilde{q}}{dt}\,\mathrm{d}t = a(T)\,\tilde{q}(T) - a(0)\,\tilde{q}(0) - \int_0^T \tilde{q}\,\frac{da(t)}{dt}\,\mathrm{d}t$$

$$\Longrightarrow \frac{\partial \mathcal{J}}{\partial q(T)}\tilde{q}(T) - (a(T)\,\tilde{q}(T) - a(0)\,\tilde{q}(0)) - \int_0^T \left[-\frac{da(t)}{dt} - \left(\frac{\partial N}{\partial q}\right)^T a\right] \cdot \tilde{q}\,\mathrm{d}t - b\,\tilde{q}(0) = 0 \quad \forall \tilde{q}$$

From the definition of $\mathcal{J}$, we deduce:

$$\frac{\partial \mathcal{J}}{\partial q(T)} = -2q(T)\frac{g \cdot g}{(q(T) \cdot q(T))^2} \qquad \Longrightarrow$$

$$\begin{aligned}
&\tilde{q}(T)\ \left[-a(T) - 2q(T)\frac{g \cdot g}{(q(T) \cdot q(T))^2}\right] \\
&+\tilde{q}(0)\ (a(0) - b) \\
&- \int_0^T \left[-\frac{da(t)}{dt} - \left(\frac{\partial N}{\partial q}\right)^T a\right] \cdot \tilde{q}\,\mathrm{d}t \\
&= 0 \quad \forall \tilde{q}
\end{aligned}$$

We deduce:

1. Adjoint equation. $\dfrac{da(t)}{dt} = -\left(\dfrac{\partial N}{\partial q}\right)^T a \quad \text{for} \quad t \in [0;T]$

2. Adjoint equation. Terminal condition. $a(T) = -2q(T)\dfrac{g \cdot g}{(q(T) \cdot q(T))^2}$

3. Compatibility condition. $a(0) = b$.

▷ **Optimality condition**: variation with respect to the control g

$$\frac{\partial \mathcal{L}}{\partial g}\tilde{g} = \lim_{\epsilon \longrightarrow 0} \frac{\mathcal{L}(q, g + \epsilon\tilde{g}, a, b) - \mathcal{L}(q, g, a, b)}{\epsilon} = 0 \qquad \forall \tilde{g}$$

Reminder: $\mathcal{L}(q,g,a,b) = \mathcal{J}(q,g) - \int_0^T a(t) \underbrace{\left[\frac{dq}{dt} - N(q)\right]}_{F(q)} \mathrm{d}t - b \underbrace{[q(0)-g]}_{F_0(q,g)}$

$$\frac{\partial \mathcal{L}}{\partial g}\tilde{g} = \frac{\partial \mathcal{J}}{\partial g}\tilde{g} + b \underbrace{\left(\frac{\partial F_0}{\partial g}\right)}_{-I} \tilde{g}\tilde{g} = 0 \qquad \forall \tilde{g}$$

$$= \left(\frac{\partial \mathcal{J}}{\partial g} + b\right)\tilde{g} = 0 \qquad \forall \tilde{g}$$

$$\Longrightarrow \frac{\partial \mathcal{J}}{\partial g} + b = 0$$

From the definition of $\mathcal{J}$, we deduce:

$$\frac{\partial \mathcal{J}}{\partial g} = \frac{2g}{q(T)\cdot q(T)} \qquad \Longrightarrow g = -b\frac{q(T)\cdot q(T)}{2}$$

We can use the compatibility equation from the adjoint equation to remove b in the optimal system. We get the following optimality system:

$$\frac{dq}{dt} = N(q) \quad ; \quad q(0) = g$$

$$-\frac{da(t)}{dt} = \left(\frac{\partial N}{\partial q}\right)^T a \quad \text{with} \quad a(T) = -2q(T)\frac{g\cdot g}{(q(T)\cdot q(T))^2}$$

$$g = -a(0)\frac{q(T)\cdot q(T)}{2}$$

Only one adjoint variable !!

Ex: $\dfrac{dq}{dt} = Lq$ with $L = \begin{bmatrix} -\dfrac{1}{\mathrm{Re}} & 0 \\ 1 & -\dfrac{3}{\mathrm{Re}} \end{bmatrix}$ *i.e.* $N(q) \triangleq Lq$, linear equation for numerical facilities, $\Longrightarrow \left(\dfrac{\partial N}{\partial q}\right)^T = L^T$.

Solution of $\dfrac{dq}{dt} = Lq$ is given by:

$$q(t) = \exp(Lt)\, q(0) \quad \Longrightarrow$$

$$G(T) = \frac{q(T)\cdot q(T)}{q(0)\cdot q(0)} = \frac{\|\exp(LT)\, q(0)\|^2}{\|q(0)\|^2} \quad \Longrightarrow$$

$$\max_{q(0)\neq 0} G(T) = \max_{q(0)\neq 0} \frac{\|\exp(LT)\, q(0)\|^2}{\|q(0)\|^2} \triangleq \|\exp(LT)\|^2$$

Scilab program: Find $G_{\mathrm{Max}} = \max_T G(T)$ for a given value of Re going from 1 to 1000. Comment on the non normality.

The optimality system now depends on time. We have to integrate forward in time the state equation and backward in time the adjoint equation. The system is solved iteratively. We consider a guess solution for the optimal forcing (g).

1. Solve the State equation

$$\frac{dq}{dt} = Lq \quad ; \quad q(0) = g$$

2. Solve the Adjoint equation

$$-\frac{da(t)}{dt} = L^T a \quad \text{with} \quad a(T) = -2q(T)\frac{g\cdot g}{(q(T)\cdot q(T))^2}$$

3. Solve the Optimality condition

$$g = -a(0)\frac{q(T)\cdot q(T)}{2}$$

$$\mathcal{J}(q,g) = \frac{g\cdot g}{q(T)\cdot q(T)} = \left.\frac{g\cdot g}{q(t)\cdot q(t)}\right|_{t=T}$$

```
//**********************************;
//* F U N C T I O N S *
//**********************************;
// State equation rhs
function [f]=StateForw(t,q,L);
f=L*q
endfunction;
//**********************************;
// Adjoint equation rhs
function [f]=AdjntBack(t,a,L);
f=-L'*a
endfunction;
//**********************************;
//* M A I N P R O G R A M *
//**********************************;
// Define system
Rey=400.0
T=200.0
L=[-1.0/Rey 0; 1 -3.0/Rey]
// Exact solution using the norm function
G_exact=(norm(expm(L*T)))^2;
// Define tolerance and initialize iterations
tol=10^(-8);
g=[rand(); rand()]; // (random initial guess)
g=g/sqrt(g'*g); // normalize
norm(g); // Check the norm of the initial condition
J=10^23; dJrel=10^23; it=0;
// Iteration loop
while (dJrel>tol)
it=it+1; Jold=J;
// forward integration of evolution eq
q0=g; [qT]=ode(q0,0,T, list(StateForw, L));
g2=g'*g; qT2=qT'*qT;
J=g2/qT2;
dJrel=abs((J-Jold)/J);
// backward integration of adjoint eq
aT=-2*qT*(g2)/qT2^2;
[a0]=ode(aT,T,0,list(AdjntBack,L));
g=-a0*(qT2/2.0);// enforce the optimality equation
end
// end of iteration loop
// print results
G=1.0/J;
it, G, G_exact
g // optimal initial condition
g=g/sqrt(g'*g) // normalize
qT // optimal response
// end of program
```

CODE 1.2: Optimal energy growth.

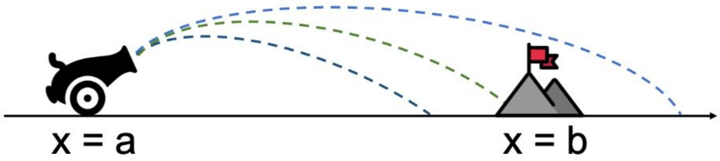


Figure 1.4: Example of shooting method.

1.11 Step #10: Example: Final target state with time-dependent forcing

The control function $g = g(t)$ is now time dependent. We consider:

$$\frac{dq}{dt} = f(q, g) \quad \text{with } g = g(t)$$
$$q(0) = q_0$$

Determine $g(t)$ such that the final state targets to P, *i.e.* $q(T) \longrightarrow P$. We introduce:

$$\mathcal{J} = \underbrace{\frac{1}{2}\left(q(T) - P\right)\left(q(T) - P\right)}_{\text{Performance term}} + \underbrace{\frac{\gamma^2}{2}\overbrace{\int_0^T (g \cdot g)\,\mathrm{d}t}^{\text{Energy control}}}_{\text{Penalization term}}$$

We consider the Lagrangian formalism and introduce:

$$\mathcal{L}(q, g, a, b) = \mathcal{J}(q, g) - \int_0^T a(t)\left[\frac{dq}{dt} - f(q, g)\right]\mathrm{d}t - b\left[q(0) - q_0\right]$$

We derive the optimality system.

▷ **Adjoint equation**: variation with respect to the state q

$$\lim_{\epsilon \longrightarrow 0} \frac{\mathcal{L}(q + \epsilon\tilde{q}, g, a, b) - \mathcal{L}(q, g, a, b)}{\epsilon} = 0 \qquad \forall\tilde{q}$$

By modifying the notations used in (1.1), we obtain:

$$\frac{\partial\mathcal{J}}{\partial q(T)}\tilde{q}(T) - \int_0^T a(t)\left(\frac{d\tilde{q}}{dt} - \frac{\partial f}{\partial q}\tilde{q}\right)\mathrm{d}t - b\,\tilde{q}(0) = 0 \qquad \forall\tilde{q}$$

Similarly as before, we modify this expression to introduce $\tilde{q}$ in factor. For that, we use the same two tricks:

1.
$$\int_0^T a(t)\,\frac{\partial f}{\partial q}\tilde{q}\,\mathrm{d}t = \int_0^T \left(\frac{\partial f}{\partial q}\right)^T a\cdot\tilde{q}\,\mathrm{d}t$$

2. Integration by parts:
$$\int_0^T a(t)\,\frac{d\tilde{q}}{dt}\,\mathrm{d}t = a(T)\,\tilde{q}(T) - a(0)\,\tilde{q}(0) - \int_0^T \tilde{q}\,\frac{da(t)}{dt}\,\mathrm{d}t$$

$$\Longrightarrow \frac{\partial \mathcal{J}}{\partial q(T)}\tilde{q}(T) - \left(a(T)\,\tilde{q}(T) - a(0)\,\tilde{q}(0)\right) - \int_0^T \left[-\frac{da(t)}{dt} - \left(\frac{\partial f}{\partial q}\right)^T a\right]\cdot\tilde{q}\,\mathrm{d}t - b\,\tilde{q}(0) = 0 \quad \forall\tilde{q}$$

From the definition of $\mathcal{J}$, we deduce:
$$\frac{\partial \mathcal{J}}{\partial q(T)} = q(T) - P \quad \Longrightarrow$$

$$\begin{aligned} &\tilde{q}(T)\ \left[-a(T) + q(T) - P\right] \\ &+\tilde{q}(0)\ \left(a(0) - b\right) \\ &-\int_0^T \left[-\frac{da(t)}{dt} - \left(\frac{\partial f}{\partial q}\right)^T a\right]\cdot\tilde{q}\,\mathrm{d}t \\ &= 0 \quad \forall\tilde{q} \end{aligned}$$

We deduce:

1. Adjoint equation. $-\dfrac{da(t)}{dt} = \left(\dfrac{\partial f}{\partial q}\right)^T a(t)$
2. Adjoint equation. Terminal condition. $a(T) = q(T) - P$
3. Compatibility condition. $a(0) = b$.

▷ **Optimality condition**: variation with respect to the control g

$$\lim_{\epsilon\longrightarrow 0}\frac{\mathcal{L}(q, g+\epsilon\tilde{g}, a, b) - \mathcal{L}(q, g, a, b)}{\epsilon} = 0 \qquad \forall\tilde{g}$$

Reminder: $\mathcal{L}(q,g,a,b) = \mathcal{J}(q,g) - \int_0^T a(t)\ \left[\frac{dq}{dt} - f(q,g)\right]\mathrm{d}t - b\ \left[q(0) - q_0\right]$

We obtain (almost immediately) that:

$$\frac{\partial \mathcal{J}}{\partial g}\tilde{g} - \int_0^T a(t)\left(-\frac{\partial f}{\partial g}\tilde{g}\right)\mathrm{d}t = 0 \qquad \forall \tilde{g}$$
$$\frac{\partial \mathcal{J}}{\partial g}\tilde{g} + \int_0^T \left(\frac{\partial f}{\partial g}\right)^T a \cdot \tilde{g}\,\mathrm{d}t = 0 \qquad \forall \tilde{g}$$

Since $\mathcal{J} = \frac{1}{2}\left(q(T)-P\right)\left(q(T)-P\right) + \frac{\gamma^2}{2}\int_0^T (g \cdot g)\,\mathrm{d}t$, we got

$$\frac{\partial \mathcal{J}}{\partial g}\tilde{g} = \gamma^2 \int_0^T g \cdot \bullet\,\mathrm{d}t \quad \text{operator notation} \quad \Longrightarrow$$

$$\gamma^2 \int_0^T g \cdot \tilde{g}\,\mathrm{d}t + + \int_0^T \left(\frac{\partial f}{\partial g}\right)^T a \cdot \tilde{g}\,\mathrm{d}t = 0 \qquad \forall \tilde{g}$$
$$\int_0^T \left(\gamma^2 g + \left(\frac{\partial f}{\partial g}\right)^T a\right) \cdot \tilde{g}\,\mathrm{d}t = 0 \qquad \forall \tilde{g}$$
$$\Longrightarrow g = -\frac{1}{\gamma^2}\left(\frac{\partial f}{\partial g}\right)^T a \quad \text{Optimality condition}$$

We get the following optimality system:

$$\frac{dq}{dt} = f(q,g) \quad ; \quad q(0) = q_0 \quad \text{State equation}$$
$$-\frac{da(t)}{dt} = \left(\frac{\partial f}{\partial q}\right)^T a \quad \text{with} \quad a(T) = q(T) - P \quad \text{Adjoint equation}$$
$$g = -\frac{1}{\gamma^2}\left(\frac{\partial f}{\partial g}\right)^T a \quad \text{Optimality condition}$$

1.12 Step #11: Example: Optimal temporal distribution of heating

Objective: Target the temperature $\theta(t)$ at time T to P.
We introduce the state variable q as

$$q(t) = \theta(t) - \theta_e$$

where θ_e is the external temperature. We assume that $\theta(0) = \theta_e$, *i.e.* $q(0) = 0$. The state equation is given by

$$\begin{aligned} \frac{dq}{dt} &= -Aq + Bg \quad \text{with } g : \text{heating depending on time} \\ q(0) &= q_0 \end{aligned}$$

The cost function is:

$$\mathcal{J} = \frac{1}{2}\left(q(T) - P\right)^2 + \frac{\gamma^2}{2}\int_0^T g^2 \, \mathrm{dt}$$

We are exactly in the framework of Sec. 1.11 with

$$f = -Aq + Bg \quad \Longrightarrow \quad \frac{\partial f}{\partial q} = -A \quad \text{and} \quad \frac{\partial f}{\partial g} = B$$

Optimality system:

$$\begin{aligned} \frac{dq}{dt} &= -Aq + Bg \quad ; \quad q(0) = q_0 \quad \text{State equation} \\ \frac{da(t)}{dt} &= A\,a(t) \quad \text{with} \quad a(T) = q(T) - P \quad \text{Adjoint equation} \\ g &= -\frac{1}{\gamma^2} B\,a \quad \text{Optimality condition} \end{aligned}$$

Analytical solution: We insert the expression of g in the equation for q. We got:

$$\frac{d\Phi}{dt} = C\Phi \quad \Longrightarrow \quad \boxed{\Phi(t) = \exp\left(Ct\right)\Phi(0)} \tag{1.2}$$

where

$$\Phi = \begin{pmatrix} q \\ a \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} -A & -\frac{1}{\gamma^2}B^2 \\ 0 & A \end{pmatrix}$$

We note

$$M(t) = Ct = \begin{pmatrix} -At & -\frac{1}{\gamma^2}B^2 t \\ 0 & At \end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ 0 & c(t) \end{pmatrix}$$

By definition of the exponent of a matrix, we have:

$$\exp(Ct) = \exp(M(t)) = \sum_{k\in\mathbb{N}} \frac{1}{k!} M^k$$

We can prove that

$$M^k = \begin{pmatrix} a^k(t) & \alpha_k(t) \\ 0 & c^k(t) \end{pmatrix}$$

where

$$\alpha_k = b\left(a^{k-1}c^0 + a^{k-2}c + \cdots + a^0 c^{k-1}\right) = b\frac{a^k - c^k}{a - c}$$

We finally deduce that:

$$\exp(M(t)) = \begin{pmatrix} \exp(a(t)) & x(t) \\ 0 & \exp(c(t)) \end{pmatrix}$$

where

$$x(t) = \frac{b\left(\exp(a(t)) - \exp(c(t))\right)}{a - c} = -\frac{1}{\gamma^2}\frac{B^2}{A}\sinh(At)$$

The solution of (1.2) is given by:

$$\Phi(t) = \begin{pmatrix} q(t) \\ a(t) \end{pmatrix} = \begin{pmatrix} \exp(-A(t)) & x(t) \\ 0 & \exp(A(t)) \end{pmatrix} \begin{pmatrix} q(0) \\ a(0) \end{pmatrix} \quad \text{i.e.}$$

$$\boxed{q(t) = \exp(-A(t))\, q(0) + x(t)a(0)} \quad \text{and} \quad \boxed{a(t) = \exp(A(t))\, a(0)}$$

Using the terminal condition $a(T) = q(T) - P$ and assuming that $q(0) = 0$, we can prove that:

$$q(t) \triangleq q_{\text{opt}}(t) = P\,\frac{B^2}{\gamma^2}\frac{\sinh(At)}{A\exp(AT) + \frac{B^2}{\gamma^2}\sinh(AT)}$$

and

$$g(t) \triangleq g_{\text{opt}}(t) = P\,\frac{AB}{\gamma^2}\frac{\exp(At)}{A\exp(AT) + \frac{B^2}{\gamma^2}\sinh(AT)}$$

Scilab program: Solve the problem for $A = B = 1$; $P = 1$; $T = 6$.

```
//**********************************;
//* F U N C T I O N S *
//**********************************;
// Returns the rhs state equation;
function [rhs]=StaEqnRHS(tloc, q, g, A, B);
rhs=-A*q+B*g;
endfunction;
//**********************************;
// Returns the rhs of the adjoint;
function [rhs]=AdjEqnRHS(tloc, a, A);
rhs=A*a
endfunction;
//**********************************;
//* M A I N P R O G R A M *
//**********************************;
// Here we give a value to the parameters
// and choose the initial conditions
A=1.0, B=1.0 // coefficients of the equation
q0=0.0 // initial condition
p=1.0 // target temperature
T=6.0 // target time
gam2=0.01 // weight of control cost gamma^2
maxiter=15 // iterations
Nt=200 // retained time samples
alpha=min(0.5,gam2) // relaxation factor
for j=1:Nt;
    t(j)=(j - 1)*T/(Nt - 1); // time grid
end;
g=0.0*t; //initialize to zero the control;
// perform iterations up to maxiter
```

CODE 1.3: Final target. Time dependent forcing. Part 1

```
// perform iterations up to maxiter
for iter=1:maxiter
// Integrate state eqn forward in time;
    q(1)=q0; // give IC;
    for j=2:Nt;
// local time interval and local IC
        t_i=t(j - 1); t_f=t(j); q_i=q(j - 1);
// local control in the time interval:
        gloc=0.5*(g(j)+g(j - 1));
// integrate forward and store solution in q_f
        [q_f]=ode(q_i,t_i,t_f,list(StaEqnRHS, gloc, A, B));
        q(j)=q_f;
    end
// compute cost of control and total cost
    g2int=0.5*g(1)^2+sum(g(2:Nt - 1)^2)+0.5*g(Nt)^2;
    Jg(iter)=0.5*gam2*(T/(Nt - 1))*g2int;
    J(iter)=0.5*(q(Nt) - p)^2+Jg(iter);
// Integrate the adjoint equations backward in time
// enforce IC (at T) for backward integration
    a(Nt)=(q(Nt) - p);
    for j=Nt-1:-1:1;
// local time interval
        t_i=t(j+1); t_f=t(j); a_i=a(j+1);
// integrate backward and store solution in a_f
        [a_f] =ode(a_i,t_i,t_f,list(AdjEqnRHS, A));
        a(j)=a_f;
    end
// Enforce optimality cond. using under-relaxation;
    g=(1 - alpha)*g+alpha*(-B/gam2)*a;
// and plot the result of the current iteration;
// Plot q(t) and g(t);
    xset("window",0);
    xtitle('State and control (all iterations)', 't', 'q (green) and g (blue
    plot2d(t',q',style=3); // style = 3 green
    plot2d(t',g',style=2); // style = 2 blue
// Plot a(t);
    xset("window",1); xtitle("Costate (all iterations)", 't', 'a');
    plot2d(t', a',style=1);
end // end of the iteration loop
pause
// Compare to exact solutions computed for q0=0;
DEN=(A*exp(A*T)+(B^2/gam2)*sinh(A*T));
q_ex=p*(B^2/gam2)*sinh(A*t)/DEN;
g_ex=p*(A*B/gam2)*exp(A*t)/DEN;
xset("window", 3); clf();
xtitle('q versus q_ex', 't', 'q');
plot2d(t',q_ex', style=-3); plot2d(t',q', style=3);
xset("window",4); clf();
xtitle('g versus g_ex', 't', 'g');
plot2d(t',g_ex',style=-2); plot2d(t', g', style=2);
// Print cost function history and convergence
for iter=2:maxiter
dJrel(iter)=abs(1.0 - J(iter - 1)/J(iter));
end
[[1:maxiter]' J dJrel]
```

Code 1.4: Final target. Time dependent forcing. Part 2

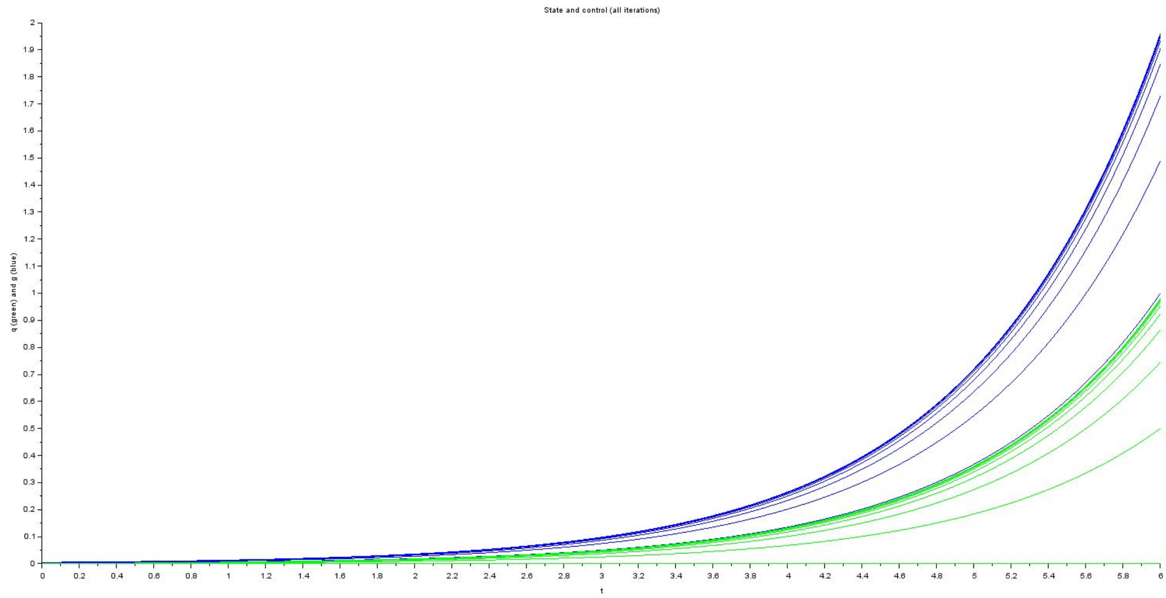


Figure 1.5: Time evolution of state and control variables at all iterations.

1.13 Step #12: Example: Feedback control of linear systems with quadratic cost functions

LQ systems: Linear Quadratic systems

$$\frac{dq}{dt} = Aq + Bg \quad \text{with} \quad q(0) = q_0 \qquad \text{State equation}$$

Determine q and g such that

$$\mathcal{J}(q, g) = \frac{1}{2} \int_0^T \left(\underbrace{q \cdot Qq}_{\text{Performance}} + \gamma^2 \underbrace{g \cdot g}_{\text{Cost control}} \right) \mathrm{d}t$$

is minimized where Q is a symmetric positive definite matrix.

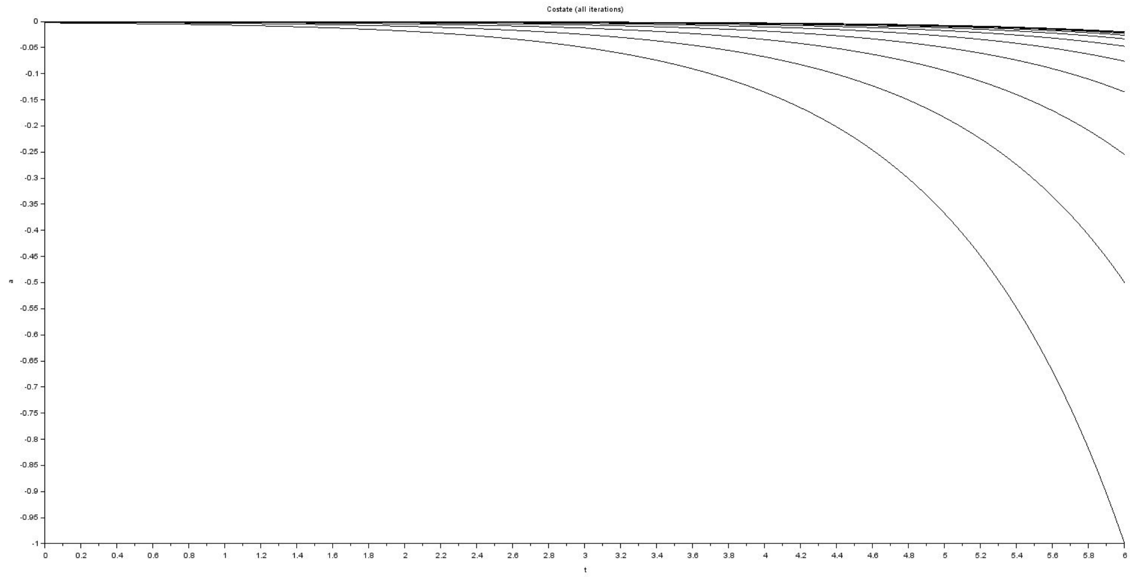


Figure 1.6: Time evolution of costate variables at all iterations.

Ex: We consider the closed-loop solution of $\frac{dq}{dt} = Aq + Bg$ with $g = -Kq$ where K is the Kalman gain. We have directly:

$$\frac{dq}{dt} = (A - BK)\,q \qquad \text{closed-loop system}$$

Scilab program: $A = \begin{bmatrix} -\frac{1}{\mathrm{Re}} & 0 \\ 1 & -\frac{3}{\mathrm{Re}} \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathrm{Re} = 100$.

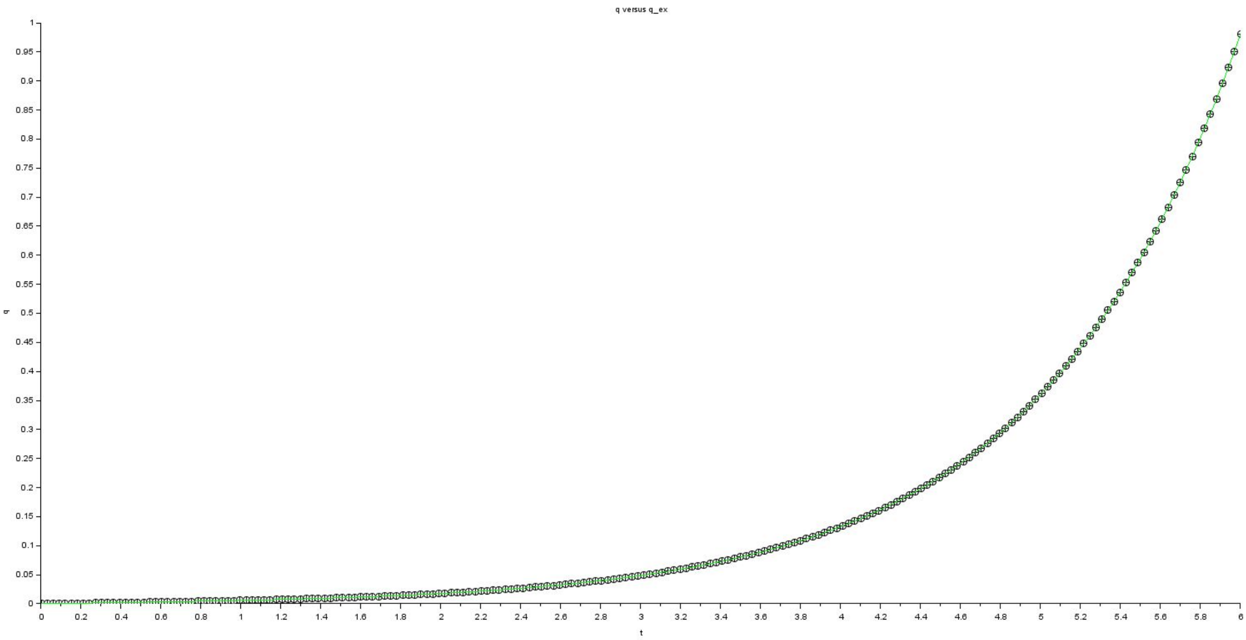


Figure 1.7: Comparison of q and q_{exact}.

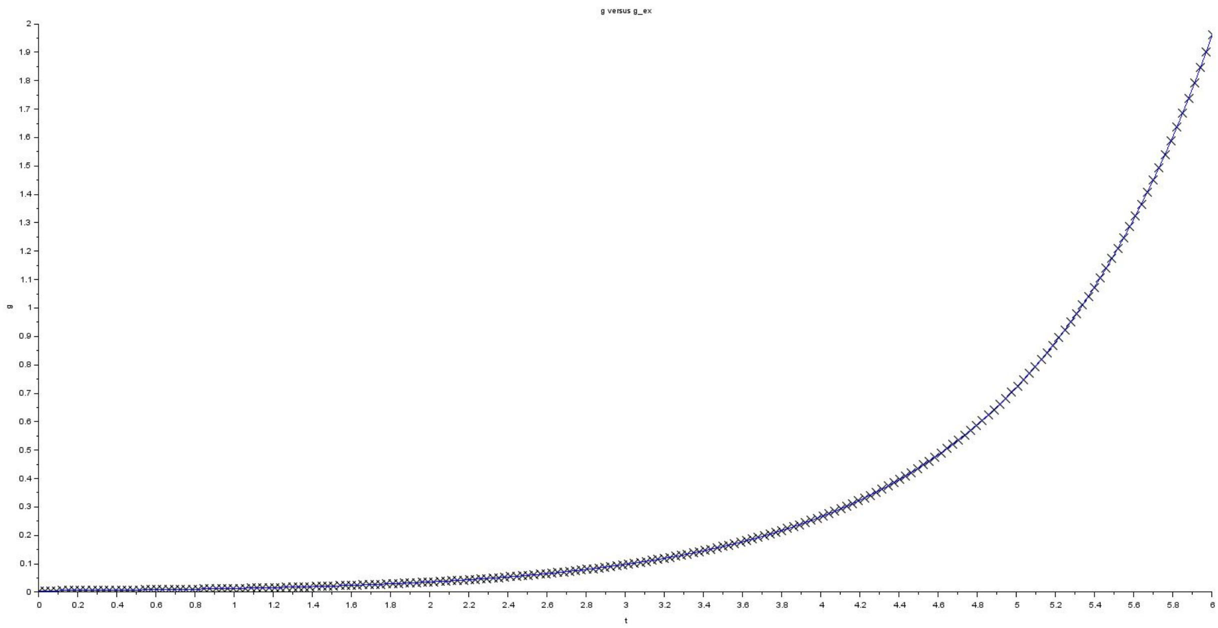


Figure 1.8: Comparison of g and g_{exact}.

```
// define system
Rey=100
A=[-1/Rey, 0; 1, -3/Rey]
B=[1, 0; 0, 1]
Q=[1, 0; 0, 1]
gam=1000
// solve Riccati equation
R=B*B'/gam^2;
X=riccati(A, R, Q, 'c', 'schur')
// compute feedback matrix
K=B'*X/gam^2
// linear modal stability of uncontrolled system
disp(spec(A))
// optimal transient energy growth of uncontrolled
// system is the L2 norm of exp(At)
Nt=200; t=linspace(0,3*Rey,Nt);
for j=1:Nt;
Gunc(j)=norm(expm(A*t(j)))^2;
end
Guncmax=max(Gunc)
// linear modal stability of controlled system
disp(spec(A-B*K))
// optimal transient energy growth of controlled
// system is the L2 norm of exp((A - BK)t)
for j=1:Nt
Gcont(j)=norm(expm((A - B*K)*t(j)))^2;
end
Gcontmax=max(Gcont)
// Plot
xtitle ( "Ricatti based feedback control" , "t" , "G" );
plot(t',Gunc,"+-");
plot(t',Gcont,"o-");
legend ( "Gunc" , "Gcont" );
//plot2d(t,[(Gunc),(Gcont)])
// end of program
```

CODE 1.5: Riccati based control.

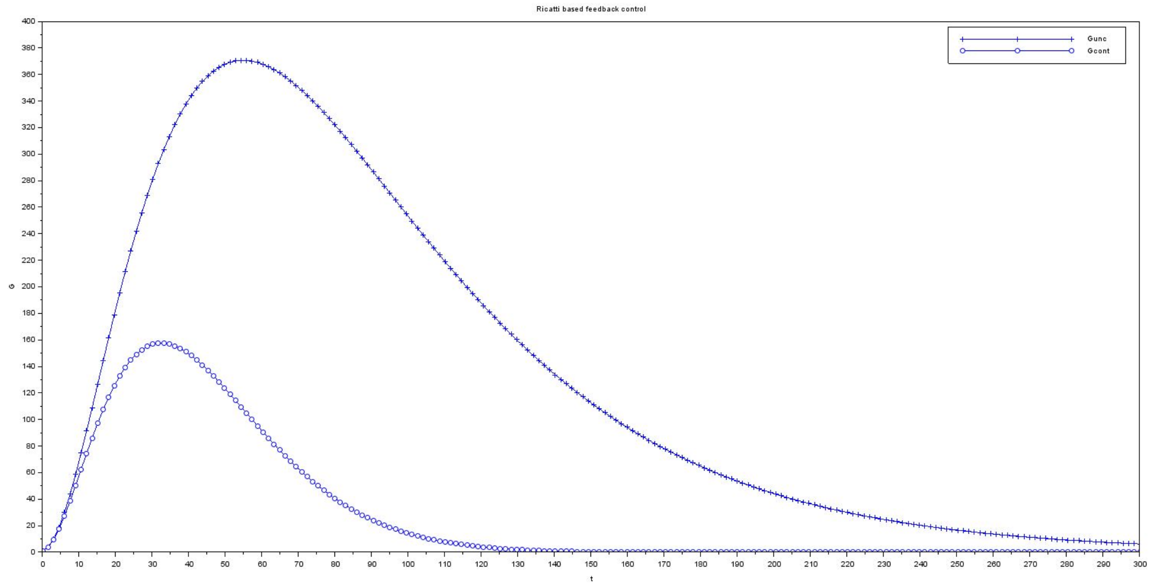


Figure 1.9: Ricatti based control. Time evolution of $G(t)$.

Optimal control

Problem of interest:

$$\text{Minimize } \mathcal{J}(q,g) \quad \text{with} \quad q \in \mathbb{R}^N \quad \text{and} \quad g \in \mathbb{R}^K \qquad \text{Cost function}$$
$$\text{Subject to } F(q,g) = 0 \qquad \text{Constraint}$$

Vocabulary:

- State variable: q
- Control variable: g
- Cost function: $\mathcal{J} = \mathcal{J}(q,g) = \mathcal{J}(q(g),g)$
- State equation: $F = F(q,g) = F(q(g),g)$

1. $q \in \mathbb{R}\,; g \in \mathbb{R}\,; F \in \mathbb{R}$
   Use the constraint equation $(F = 0)$ to eliminate a variable.

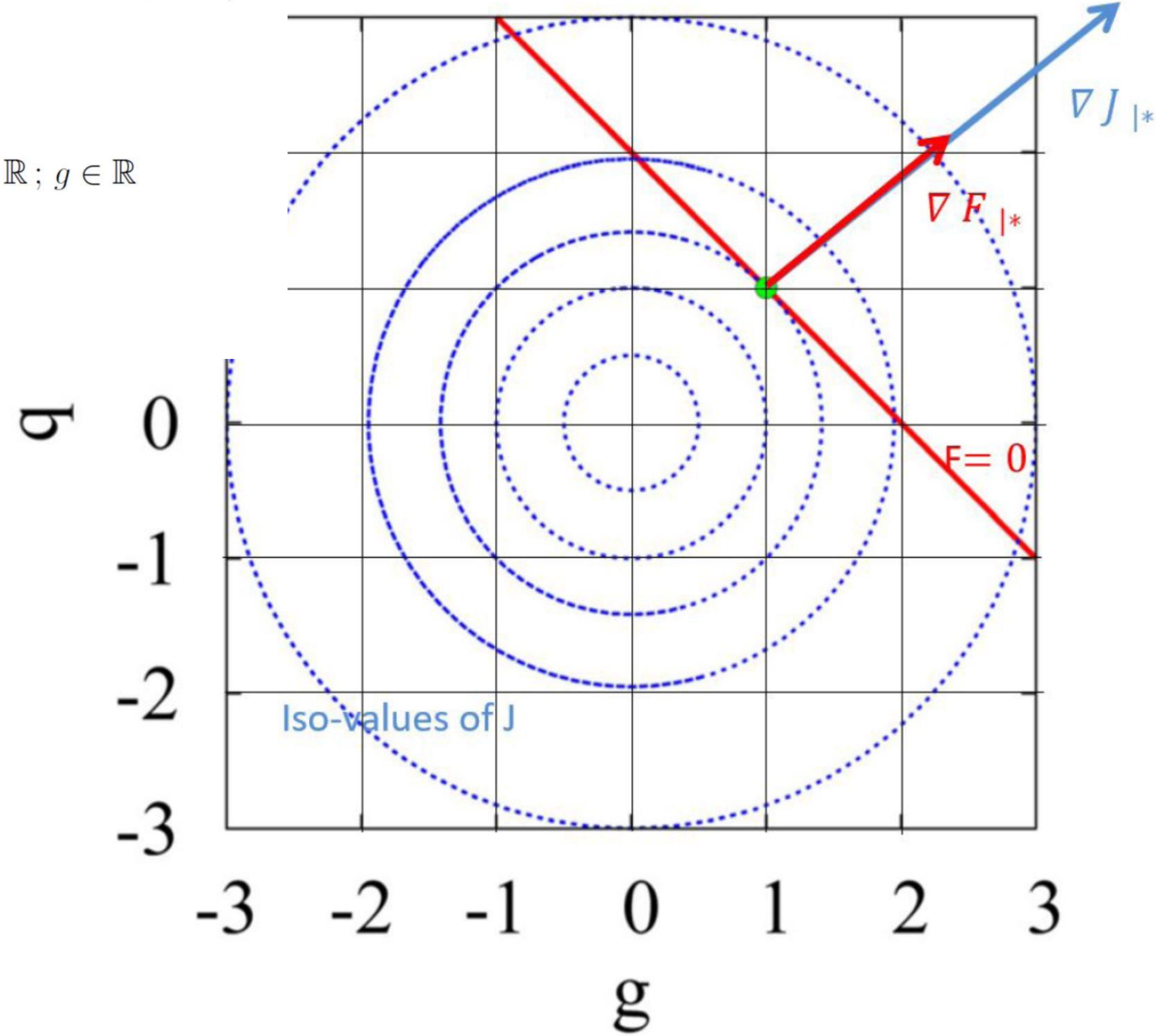
2. $q \in \mathbb{R}\,; g \in \mathbb{R}\,; F \in \mathbb{R}$
   Determine the **total derivative** $\dfrac{D\mathcal{J}}{Dg}$ through the estimation of the **sensitivity** $\dfrac{dq}{dg}$.
   $\Longrightarrow$ Differentiate $F = 0$

3. $\mathbf{q} \in \mathbb{R}^N\,; \mathbf{g} \in \mathbb{R}^K\,; \mathbf{F} \in \mathbb{R}^N$
   Determine the vectorial **total derivative** $\dfrac{D\mathcal{J}}{D\mathbf{g}}$ through the estimation of the **sensitivity**.

Introductory example: From Cossu (Applied Mechanics Review, 2014):

$$\text{Minimize } \mathcal{J}(q,g) = q^2 + g^2 \quad \text{with} \quad q \in \mathbb{R}\,;\, g \in \mathbb{R}$$
$$\text{Subject to } F(q,g) = q + g - 2 = 0$$

Unconstrained minimum: $(0,0)$
Constrained minimum: $(1,1)$ (graphic reading)



Step #1: Intuitive method: Eliminate a variable from $F = 0$

$$F = 0 \Longrightarrow F = q + g - 2 = 0 \Longrightarrow q = 2 - g \Longrightarrow \mathcal{J}(q,g) = (2-g)^2 + g^2 = 2g^2 - 4g + 4$$

$$\frac{D\mathcal{J}}{Dg} = 4(g-1) = 0 \text{ for } g^* = 1 \Longrightarrow q^* = 1 \Longrightarrow \mathcal{J}^* = 2$$

$$\frac{D^2\mathcal{J}}{Dg^2} = 4 > 0 \Longrightarrow (1,1) \text{ is a minimum}$$

What can be done if we cannot resolve explicitly $F = 0$ *i.e.* determine $q(g)$?

Step #2: Determination total derivative through the sensitivity $\frac{dq}{dg}$

$$\mathcal{J}(q(g),g) \stackrel{\text{Chain rule}}{\Longrightarrow} \frac{D\mathcal{J}}{Dg} = \underbrace{\frac{\partial \mathcal{J}}{\partial q}}_{2q} \frac{dq}{dg} + \underbrace{\frac{\partial \mathcal{J}}{\partial g}}_{2g} = 0$$

How to determine $\frac{dq}{dg}$? We differentiate $F = 0 \Longrightarrow$

$$dF = \frac{\partial F}{\partial q} dq + \frac{\partial F}{\partial g} dg$$

$$\frac{dq}{dg} = -\left(\frac{\partial F}{\partial q}\right)^{-1} \frac{\partial F}{\partial g}$$

$$\frac{D\mathcal{J}}{Dg} = \underbrace{\frac{\partial \mathcal{J}}{\partial q}}_{2q} \frac{dq}{dg} + \underbrace{\frac{\partial \mathcal{J}}{\partial g}}_{2g} = 0$$

$$\frac{dq}{dg} = -\left(\frac{\partial F}{\partial q}\right)^{-1} \frac{\partial F}{\partial g}$$

Here, $\frac{\partial F}{\partial q} = 1$ and $\frac{\partial F}{\partial g} = 1 \Longrightarrow \frac{dq}{dg} = -1 \Longrightarrow \frac{D\mathcal{J}}{Dg} = 2(g-q) \Longrightarrow$
$\frac{D\mathcal{J}}{Dg} = 0$ for $g = q$.

Since $F(q,g) = q+g-2 = 0 \Longrightarrow 2q = 2 \Longrightarrow q^* = 1 \quad ; \quad g^* = 1 \quad ; \quad \mathcal{J}^* = 2.$

Step #3: Determination vectorial total derivative $\frac{D\mathcal{J}}{D\mathbf{g}}$ through the sensitivities

We suppose $\mathbf{q} \in \mathbb{R}^N; \mathbf{F} \in \mathbb{R}^N; \mathbf{g} \in \mathbb{R}^K$ *i.e.* $\mathbf{g} = \sum_{k=1}^{K} g_k \mathbf{e}_k$ with $\mathbf{e}_k$ vectors of an orthonormal basis.

Total derivative:

$$\frac{D\mathcal{J}}{D\mathbf{g}} = \begin{pmatrix} \frac{D\mathcal{J}}{Dg_1} \\ \vdots \\ \frac{D\mathcal{J}}{Dg_K} \end{pmatrix} \in \mathbb{R}^K$$

i) Naive approximation of $\dfrac{D\mathcal{J}}{Dg_k}, k = 1, \cdots, K$ by finite differences

$$\frac{D\mathcal{J}}{Dg_k} \approx \frac{\mathcal{J}\left(\mathbf{q}\left(\mathbf{g} + \Delta g_k \mathbf{e}_k\right), \mathbf{g} + \Delta g_k \mathbf{e}_k\right) - \mathcal{J}(\mathbf{q}(\mathbf{g}), \mathbf{g})}{\Delta g_k}$$

with Δg_k small increment.

Drawbacks:

- $K+1$ resolution of $\mathbf{F} = \mathbf{0}$ are needed ; costly if solving $\mathbf{F} = \mathbf{0}$ is costly.
- prone to numerical inaccuracy: how to choose Δg_k?

ii) Generalization of the previous approach: sensitivity equations

$$\frac{D\mathcal{J}}{Dg_k} = \sum_{i=1}^{N} \frac{\partial \mathcal{J}}{\partial q_i}\frac{dq_i}{dg_k} + \frac{\partial \mathcal{J}}{\partial g_k}$$
$$= \frac{\partial \mathcal{J}}{\partial \mathbf{q}} \cdot \frac{d\mathbf{q}}{dg_k} + \frac{\partial \mathcal{J}}{\partial g_k}$$

with

$$\frac{\partial \mathcal{J}}{\partial \mathbf{q}} = \begin{pmatrix} \frac{\partial \mathcal{J}}{\partial q_1} \\ \vdots \\ \frac{\partial \mathcal{J}}{\partial q_N} \end{pmatrix} \in \mathbb{R}^N \quad \text{and} \quad \frac{d\mathbf{q}}{dg_k} = \begin{pmatrix} \frac{dq_1}{dg_k} \\ \vdots \\ \frac{dq_N}{dg_k} \end{pmatrix} \in \mathbb{R}^N$$

We differentiate $\mathbf{F}(\mathbf{q},\mathbf{g}) = 0 \Longrightarrow \dfrac{D\mathbf{F}}{Dg_k} = 0, \; \forall k = 1, \cdots, K \quad \Longrightarrow$

$$\frac{\partial \mathbf{F}}{\partial \mathbf{q}} \frac{d\mathbf{q}}{dg_k} + \frac{\partial \mathbf{F}}{\partial g_k} = 0 \quad \text{sensitivity eqs.} \quad \text{with} \quad \frac{\partial \mathbf{F}}{\partial \mathbf{q}} = \begin{pmatrix} \frac{\partial F_1}{\partial q_1}, \cdots, \frac{\partial F_1}{\partial q_N} \\ \vdots \\ \frac{\partial F_N}{\partial q_1}, \cdots, \frac{\partial F_N}{\partial q_N} \end{pmatrix} \text{Jacobian matrix}$$

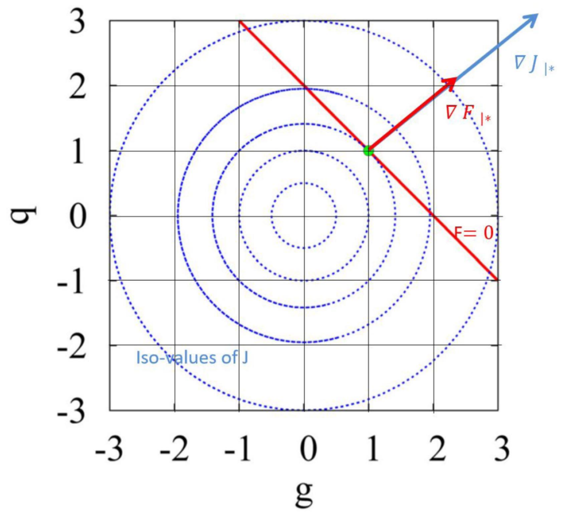
$$\boxed{\frac{d\mathbf{q}}{dg_k} = -\left(\frac{\partial \mathbf{F}}{\partial \mathbf{q}}\right)^{-1} \frac{\partial \mathbf{F}}{\partial g_k}}$$

This is the **sensitivity equations** method. $\dfrac{d\mathbf{q}}{dg_k}$ is obtained as a solution of a linear system of equations of size $N \times N$ (costly).

Step #4: Intuition of the variational formulation

At the minimum of F, $F = 0$ is tangent to the iso-value of the minimum.

We have:



$$\nabla F|_* \propto \nabla \mathcal{J}|_* \quad \text{and} \quad F = 0 \Longrightarrow$$

$$\exists a \quad \text{s.t.} \quad \begin{pmatrix} \dfrac{\partial \mathcal{J}}{\partial q} \\ \\ \dfrac{\partial \mathcal{J}}{\partial g} \end{pmatrix} = a \begin{pmatrix} \dfrac{\partial F}{\partial q} \\ \\ \dfrac{\partial F}{\partial g} \end{pmatrix} \Longrightarrow \quad \text{Optimality system}$$

$$\frac{\partial \mathcal{J}}{\partial q} - a\frac{\partial F}{\partial q} = 0 \quad \text{Adjoint equation}$$

$$\frac{\partial \mathcal{J}}{\partial g} - a\frac{\partial F}{\partial g} = 0 \quad \text{Optimlity condition}$$

$$F = 0 \quad \text{Constraint}$$

Lagrange noticed that this system of equations corresponded to the optimal system that could be written for an augmented Lagrangian defined by:

$$\mathcal{L}(q, g, a) = \mathcal{J} - aF \quad \text{with}$$

considering the variables q, g, a as independent.

Vocabulary:

- Lagrangian or augmented cost function: $\mathcal{L}$
- Co-state or Lagrange multiplier: a

Ex : Cossu (AMR, 2014). In $\mathbb{R}$

$$\frac{\partial \mathcal{J}}{\partial q} = 2q \quad ; \quad \frac{\partial \mathcal{J}}{\partial g} = 2g \quad ; \quad \frac{\partial F}{\partial q} = 1q \quad ; \quad \frac{\partial F}{\partial g} = 1q \quad ;$$

Optimality system : $2q - a = 0 \quad ; \quad 2g - a = 0 \quad ; \quad q + g - 2 = 0$

Eliminate a : $a = 2q = 2g \Longrightarrow q = g \Longrightarrow q = g = 1$

Solution : $(q, g, a)_* = (1, 1, 2) \quad ; \quad \mathcal{J}_* = 2$

Step #5: Generalization of the variational formulation. Inner product in $\mathbb{R}^N$

We introduce a Lagrange multiplier a_j for each component F_j $(j = 1, \cdots, N)$. we have:

$$\mathcal{L}(\mathbf{q}, \mathbf{g}, \mathbf{a}) = \mathcal{J}(\mathbf{q}, \mathbf{g}) - \sum_{j=1}^{N} a_j F_j(\mathbf{q}, \mathbf{g}) = \mathcal{J}(\mathbf{q}, \mathbf{g}) - \mathbf{a} \cdot \mathbf{F}(\mathbf{q}, \mathbf{g})$$

Optimality is obtained by considering that the variables $\mathbf{q}, \mathbf{g}$ and $\mathbf{a}$ are independent, *i.e.*

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} = 0 \quad ; \quad \frac{\partial \mathcal{L}}{\partial \mathbf{g}} = 0 \quad ; \quad \frac{\partial \mathcal{L}}{\partial \mathbf{a}} = 0$$

We obtain component by component: $2N + K$ equations

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\partial \mathcal{J}}{\partial q_i} - \sum_j a_j \frac{F_j}{\partial q_i} = 0 \qquad i = 1, \cdots, N$$

$$\frac{\partial \mathcal{L}}{\partial g_k} = \frac{\partial \mathcal{J}}{\partial g_k} - \sum_j a_j \frac{F_j}{\partial g_k} = 0 \qquad k = 1, \cdots, K$$

$$\frac{\partial \mathcal{L}}{\partial a_i} = -F_i = 0 \qquad i = 1, \cdots, N$$

Rk: $\displaystyle\sum_j a_j \frac{F_j}{\partial q_i} = \sum_i a_i \frac{F_i}{\partial q_j} \Longrightarrow$ Optimality system in vectorial notation:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} = 0 \quad \Longrightarrow \quad \left(\frac{\partial \mathbf{F}}{\partial \mathbf{q}}\right)^T \mathbf{a} = \frac{\partial \mathcal{J}}{\partial \mathbf{q}} \qquad \text{Adjoint equation}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{g}} = 0 \quad \Longrightarrow \quad \left(\frac{\partial \mathbf{F}}{\partial \mathbf{g}}\right)^T \mathbf{a} = \frac{\partial \mathcal{J}}{\partial \mathbf{g}} \qquad \text{Optimality condition}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{a}} = 0 \quad \Longrightarrow \quad \mathbf{F} = \mathbf{0} \qquad \text{State equation}$$

Step #6: Generalization to function gradients ($\mathbb{R}^N$)

We consider the first order variation $\delta\mathcal{L}$ introduced by small variations $\delta\mathbf{q}$, $\delta\mathbf{g}$ and $\delta\mathbf{a}$. At the optimum, we have:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} = 0 \quad ; \quad \frac{\partial \mathcal{L}}{\partial \mathbf{g}} = 0 \quad ; \quad \frac{\partial \mathcal{L}}{\partial \mathbf{a}} = 0$$

By definition, the variation $\delta\mathcal{L}$ induced by a small variation $\delta\mathbf{a} = \epsilon\tilde{\mathbf{a}}$ with ϵ a small parameter, is given by the <u>directional derivative</u>:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{a}}\tilde{\mathbf{a}} \triangleq \lim_{\epsilon \longrightarrow 0} \frac{\mathcal{L}(\mathbf{q}, \mathbf{g}, \mathbf{a} + \epsilon\tilde{\mathbf{a}}) - \mathcal{L}(\mathbf{q}, \mathbf{g}, \mathbf{a})}{\epsilon}$$

<u>Reminder</u>: $\mathcal{L}(\mathbf{q}, \mathbf{g}, \mathbf{a}) = \mathcal{J}(\mathbf{q}, \mathbf{g}) - \sum_{j=1}^{N} a_j F_j(\mathbf{q}, \mathbf{g}) = \mathcal{J}(\mathbf{q}, \mathbf{g}) - \mathbf{a} \cdot \mathbf{F}(\mathbf{q}, \mathbf{g})$

- ▷ Variation with respect to the co-state $\mathbf{a}$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \mathbf{a}}\tilde{\mathbf{a}} &= \lim_{\epsilon \longrightarrow 0} -\frac{1}{\epsilon}\left[(\mathbf{a}+\epsilon\tilde{\mathbf{a}})\cdot\mathbf{F} - \mathbf{a}\cdot\mathbf{F}\right] \\ &= -\tilde{\mathbf{a}}\cdot\mathbf{F} = 0 \quad \forall\tilde{\mathbf{a}} \quad \Longrightarrow \mathbf{F} = \mathbf{0} \quad \text{State equation}\end{aligned}$$

▷ Variation with respect to the state $\mathbf{q}$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}}\tilde{\mathbf{q}} = \lim_{\epsilon \longrightarrow 0} \frac{\mathcal{L}(\mathbf{q}+\epsilon\tilde{\mathbf{q}},\mathbf{g},\mathbf{a}) - \mathcal{L}(\mathbf{q},\mathbf{g},\mathbf{a})}{\epsilon}$$

$$\begin{aligned}\mathcal{L}(\mathbf{q}+\epsilon\tilde{\mathbf{q}},\mathbf{g},\mathbf{a}) &= \mathcal{J}(\mathbf{q}+\epsilon\tilde{\mathbf{q}},\mathbf{g}) - \mathbf{a}\cdot\mathbf{F}(\mathbf{q}+\epsilon\tilde{\mathbf{q}},\mathbf{g})\\ \mathcal{L}(\mathbf{q},\mathbf{g},\mathbf{a}) &= \mathcal{J}(\mathbf{q},\mathbf{g}) - \mathbf{a}\cdot\mathbf{F}(\mathbf{q},\mathbf{g})\end{aligned}$$

We perform Taylor developments to order 2.

$$\begin{aligned}\mathcal{J}(\mathbf{q}+\epsilon\tilde{\mathbf{q}},\mathbf{g}) &= \mathcal{J}(\mathbf{q},\mathbf{g}) + \epsilon\frac{\partial \mathcal{J}}{\partial \mathbf{q}}\cdot\tilde{\mathbf{q}} + \mathcal{O}(\epsilon^2)\\ \mathbf{F}(\mathbf{q}+\epsilon\tilde{\mathbf{q}},\mathbf{g}) &= \mathbf{F}(\mathbf{q},\mathbf{g}) + \epsilon\frac{\partial \mathbf{F}}{\partial \mathbf{q}}\tilde{\mathbf{q}} + \mathcal{O}(\epsilon^2)\end{aligned}$$

$$\Longrightarrow \frac{\partial \mathcal{L}}{\partial \mathbf{q}}\tilde{\mathbf{q}} = \frac{\partial \mathcal{J}}{\partial \mathbf{q}}\cdot\tilde{\mathbf{q}} - \mathbf{a}\cdot\left(\frac{\partial \mathbf{F}}{\partial \mathbf{q}}\right)\tilde{\mathbf{q}}$$

Rk: $\mathbf{u} \cdot A\mathbf{v} = A^T\mathbf{u} \cdot \mathbf{v} \quad \Longrightarrow$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}}\tilde{\mathbf{q}} = \frac{\partial \mathcal{J}}{\partial \mathbf{q}} \cdot \tilde{\mathbf{q}} - \left(\frac{\partial \mathbf{F}}{\partial \mathbf{q}}\right)^T \mathbf{a} \cdot \tilde{\mathbf{q}}$$

$$= \left[\frac{\partial \mathcal{J}}{\partial \mathbf{q}} - \left(\frac{\partial \mathbf{F}}{\partial \mathbf{q}}\right)^T \mathbf{a}\right] \cdot \tilde{\mathbf{q}} = 0 \quad \forall \tilde{\mathbf{q}} \quad \Longrightarrow \frac{\partial \mathcal{J}}{\partial \mathbf{q}} - \left(\frac{\partial \mathbf{F}}{\partial \mathbf{q}}\right)^T \mathbf{a} = \mathbf{0} \quad \text{Adjoint equation}$$

▷ Variation with respect to the control $\mathbf{g}$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{g}}\tilde{\mathbf{g}} = \lim_{\epsilon \longrightarrow 0} \frac{\mathcal{L}(\mathbf{q}, \mathbf{g} + \epsilon \tilde{\mathbf{g}}, \mathbf{a}) - \mathcal{L}(\mathbf{q}, \mathbf{g}, \mathbf{a})}{\epsilon}$$

$$\frac{\partial \mathcal{J}}{\partial \mathbf{g}} \cdot \tilde{\mathbf{g}} - \mathbf{a} \cdot \left(\frac{\partial \mathbf{F}}{\partial \mathbf{g}}\right) \tilde{\mathbf{g}}$$

$$= \left[\frac{\partial \mathcal{J}}{\partial \mathbf{g}} - \left(\frac{\partial \mathbf{F}}{\partial \mathbf{g}}\right)^T \mathbf{a}\right] \cdot \tilde{\mathbf{g}} = 0 \quad \forall \tilde{\mathbf{g}} \quad \Longrightarrow \frac{\partial \mathcal{J}}{\partial \mathbf{g}} - \left(\frac{\partial \mathbf{F}}{\partial \mathbf{g}}\right)^T \mathbf{a} = \mathbf{0} \quad \text{Optimality conditio}$$

In summary, we obtain the optimality system composed of:

$$\mathbf{F} = \mathbf{0} \quad \text{State equation}$$

$$\frac{\partial \mathcal{J}}{\partial \mathbf{q}} - \left(\frac{\partial \mathbf{F}}{\partial \mathbf{q}}\right)^T \mathbf{a} = \mathbf{0} \quad \text{Adjoint equation}$$

$$\frac{\partial \mathcal{J}}{\partial \mathbf{g}} - \left(\frac{\partial \mathbf{F}}{\partial \mathbf{g}}\right)^T \mathbf{a} = \mathbf{0} \quad \text{Optimality condition}$$

This is a system of coupled equations that are solved by *one shot method* or *iterative method*.

Step #7: Gradient method based on the adjoint equation

Objective: Show that the use of adjoint equations decreases the number of operations needed to calculate the total derivative.

We showed previously that:

$$\frac{D\mathcal{J}}{Dg_k} = \frac{\partial \mathcal{J}}{\partial \mathbf{q}} \cdot \frac{d\mathbf{q}}{dg_k} + \frac{\partial \mathcal{J}}{\partial g_k}$$
$$= \sum_{i=1}^{N} \frac{\partial \mathcal{J}}{\partial q_i} \frac{dq_i}{dg_k} + \frac{\partial \mathcal{J}}{\partial g_k}$$

We search to simplify this expression.

1. Introduction of the adjoint equation $\dfrac{\partial \mathcal{J}}{\partial q_i} = \sum_j a_j \dfrac{\partial F_j}{\partial q_i} \Longrightarrow$

$$\frac{D\mathcal{J}}{Dg_k} = \sum_i \sum_j a_j \frac{\partial F_j}{\partial q_i} \frac{dq_i}{dg_k} + \frac{\partial \mathcal{J}}{\partial g_k}$$

2. Let's differentiate $\mathbf{F}(\mathbf{q}, \mathbf{g}) = 0 \Longrightarrow \dfrac{D\mathbf{F}}{Dg_k} = 0, \forall k \Longrightarrow \dfrac{\partial \mathbf{F}}{\partial \mathbf{q}} \dfrac{d\mathbf{q}}{dg_k} + \dfrac{\partial \mathbf{F}}{\partial g_k} = 0$

   We deduce for the j-th component: $\displaystyle\sum_i \frac{\partial F_j}{\partial q_i} \frac{dq_i}{dg_k} = -\frac{\partial F_j}{\partial g_k}.$

Finally, we obtain: $$\frac{D\mathcal{J}}{Dg_k} = -\sum_j a_j \frac{\partial F_j}{\partial g_k} + \frac{\partial \mathcal{J}}{\partial g_k}$$

The vectorial expression is

$$\boxed{\frac{D\mathcal{J}}{D\mathbf{g}} = -\left(\frac{\partial \mathbf{F}}{\partial \mathbf{g}}\right)^T \mathbf{a} + \frac{\partial \mathcal{J}}{\partial \mathbf{g}}} \quad \text{with} \quad \boxed{\left(\frac{\partial \mathbf{F}}{\partial \mathbf{q}}\right)^T \mathbf{a} = \frac{\partial \mathcal{J}}{\partial \mathbf{q}}} \quad \text{Adjoint eq.}$$

If we know $\dfrac{\partial \mathcal{J}}{\partial \mathbf{q}}$, $\dfrac{\partial \mathcal{J}}{\partial \mathbf{g}}$, $\dfrac{\partial \mathbf{F}}{\partial \mathbf{q}}$ and $\dfrac{\partial \mathbf{F}}{\partial \mathbf{g}}$, then the cost of determining the total derivative is given by the cost of solving the adjoint equation. This cost corresponds to the solution of one linear system of equations of size N. Then, this is much less costly to use the adjoint equations for determining the total derivative.

Step #8: Example: Optimal amplification of forcing with a steady linear system

Consider $\quad Lu + f = 0$

We look for f (steady) that maximizes the energy amplification defined as

$$R = \frac{u \cdot u}{f \cdot f}$$

Rk 1: Maximizing R is equivalent to minimizing $1/R$.
Rk 2: Formally, we have:

$$u = -L^{-1} f \quad \Longrightarrow \quad R = \frac{L^{-1} f \cdot L^{-1} f}{f \cdot f} = \frac{\|L^{-1} f\|^2}{\|f\|^2}$$

$$\max_{f \neq 0} R = \max_{f \neq 0} \frac{\|L^{-1} f\|^2}{\|f\|^2} \triangleq \|L^{-1}\|^2$$

For solving the maximization problem over f, we use the **optimal control** approach for:

$$q \equiv u \quad ; \quad g \equiv f \quad \text{and} \quad K = N$$

$$F(q,g) = Lq + g \qquad \text{State equation}$$

$$\mathcal{J}(q,g) = \frac{g \cdot g}{q \cdot q} = \frac{1}{R} \qquad \text{Cost function} \quad (\text{minimization of } \mathcal{J} \text{ for maximization of } R)$$

Optimality system:

$$F = 0 \quad \text{State equation}$$

$$\left(\frac{\partial F}{\partial q}\right)^T a = \frac{\partial \mathcal{J}}{\partial q} \quad \text{Adjoint equation}$$

$$\left(\frac{\partial F}{\partial g}\right)^T a = \frac{\partial \mathcal{J}}{\partial g} \quad \text{Optimality condition}$$

$$\frac{\partial F}{\partial q} = L \quad ; \quad \frac{\partial F}{\partial g} = I \quad ; \quad \frac{\partial \mathcal{J}}{\partial q} = -2q\frac{g \cdot g}{(q \cdot q)^2} \quad ; \quad \frac{\partial \mathcal{J}}{\partial q} = \frac{2g}{q \cdot q} \quad ;$$

$$F = 0 \quad \text{State equation}$$

$$\left(\frac{\partial F}{\partial q}\right)^T a = \frac{\partial \mathcal{J}}{\partial q} \quad \text{Adjoint equation}$$

$$\left(\frac{\partial F}{\partial g}\right)^T a = \frac{\partial \mathcal{J}}{\partial g} \quad \text{Optimality condition}$$

$$\frac{\partial F}{\partial q} = L \qquad \frac{\partial F}{\partial g} = I$$

$$\frac{\partial \mathcal{J}}{\partial q} = -2q\frac{g \cdot g}{(q \cdot q)^2}$$

$$\frac{\partial \mathcal{J}}{\partial q} = \frac{2g}{q \cdot q}$$

$$\text{Adjoint equation} \Longrightarrow \quad L^T a = -2q\frac{g \cdot g}{(q \cdot q)^2} \quad \text{Dim. } N$$

$$\text{Optimality condition} \Longrightarrow \quad a = \frac{2g}{q \cdot q} \Longrightarrow g = \frac{1}{2}a\,(q \cdot q) \quad \text{Dim. } K$$

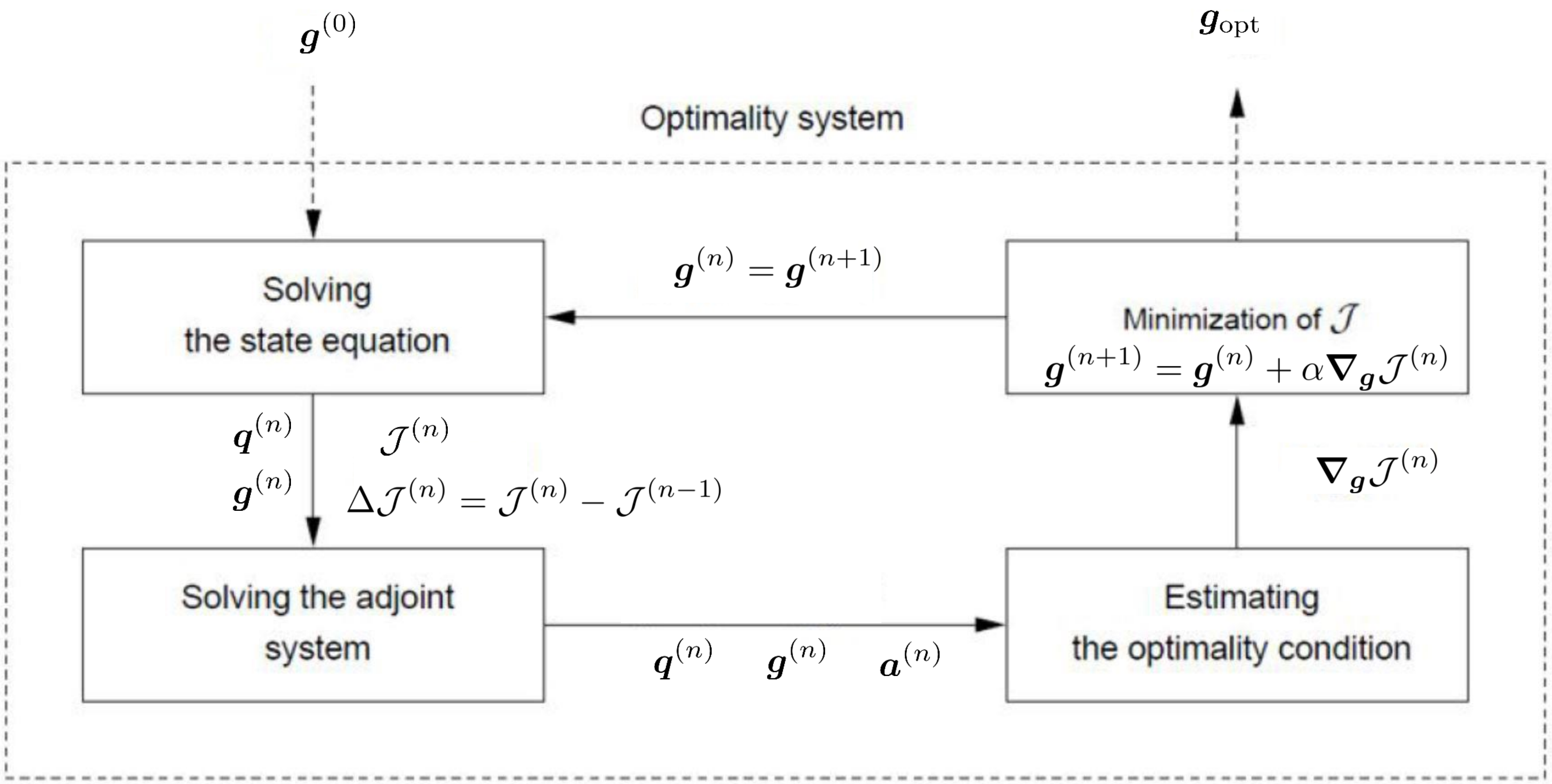
$$\text{State equation} \Longrightarrow \quad Lq + g = 0 \quad \text{Dim. } N$$

Since $K = N$, the size of the optimality system is $2N + K = 3N$.

Iterative resolution of the optimality system

1. Given the n-th guess for the optimal forcing $g^{(n)}$, compute $q^{(n)}$ by solving the state equation:
$$Lq^{(n)} = -g^{(n)}$$
2. Compute $\mathcal{J}$ and $\Delta\mathcal{J}^{(n)}$ the increment of $\mathcal{J}$ between two iterations.
3. Compute the adjoint state $a^{(n)}$ solving the adjoint equation.
4. Determine $g^{(n+1)}$ using the optimality condition and go to (1).

Iterative resolution of the optimality system



Ex: $L = \begin{bmatrix} -\dfrac{1}{\mathrm{Re}} & 0 \\ 1 & -\dfrac{3}{\mathrm{Re}} \end{bmatrix}$

Rk 1: We define as $L^* = L^T$, the adjoint matrix of L. L is called a **non normal** matrix, since $LL^* \neq L^*L$.
Rk 2: In an Hermitian space, a matrix is normal if and only if it is diagonalizable in an orthonormal basis.

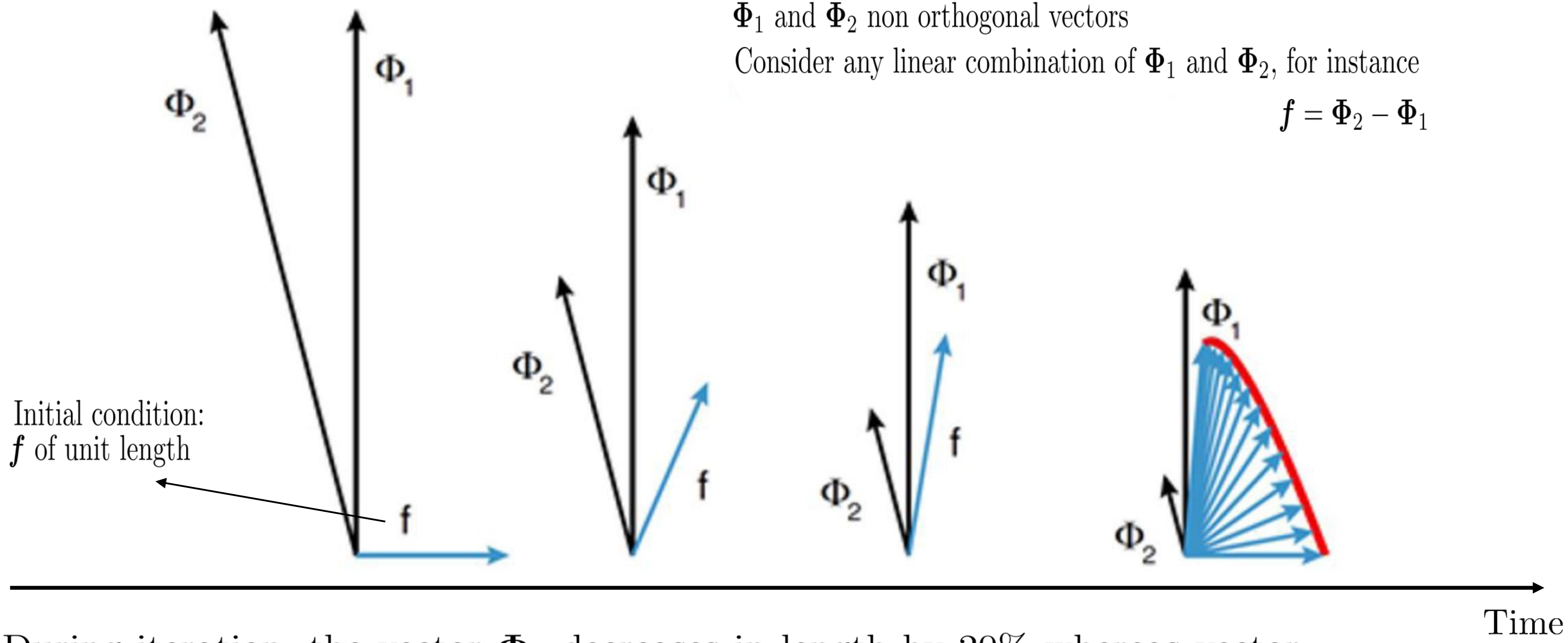
Rk 3: $\lambda(L) = \{-\dfrac{1}{\mathrm{Re}}; -\dfrac{3}{\mathrm{Re}}\}$. At least, one eigenvalue is strictly negative, then it means the linearized system based on L is stable. There is a decay at long time evolution of the solution.

Non normal transient growth (after Schmid, ARFM 2007)

$\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ non orthogonal vectors

Consider any linear combination of $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$, for instance

$$f = \mathbf{\Phi}_2 - \mathbf{\Phi}_1$$



During iteration, the vector $\mathbf{\Phi}_1$ decreases in length by 20% whereas vector $\mathbf{\Phi}_2$ shrinks by 50%.

The vector f transiently increases in length and aligns itself with $\mathbf{\Phi}_1$, the least stable eigenvector.

f decreases to zero in the large time limit.

Scilab program:

The optimality system is solved iteratively.

We consider a guess solution for the optimal forcing (g).

1. Solve the State equation

$$Lq + g = 0 \quad \Longrightarrow \quad q = -L^{-1}g$$

2. Solve the Adjoint equation

$$L^T a = -2q\frac{g \cdot g}{(q \cdot q)^2} \quad \Longrightarrow \quad a$$

2. Solve the Adjoint equation

$$L^T a = -2q \frac{g \cdot g}{(q \cdot q)^2} \quad \Longrightarrow \quad a$$

3. Solve the Optimality condition

$$g = \frac{1}{2} a \, (q \cdot q)$$

$$\mathcal{J}(q, g) = \frac{g \cdot g}{q \cdot q}$$

Optimal_Amplification_Forcing_Steady_Linear_System.sce

```
// Define system
// Reynolds number
Rey=40.0
// L matrix (L is non normal)
L=[-1.0/Rey 0; 1 -3.0/Rey]
// Exact solution using the norm function
R_exact=(norm(inv(-L)))^2
// Define tolerance and initialize iterations
// Tolerance for convergence
tol=10^(-8);
// Initialize control (random)
g=[rand(); rand()];
// Normalize g (optional)
//g=g/norm(g);
// Initialize J
J=10^23;
// Initialize dJrel = (J^{(n+1)}-J^{(n)})/J^{(n)} : relative variation
dJrel=10^23
// Initialize it : iteration number
it=0;
//
```

```
// Iteration loop
// While not converged
while (dJrel>tol)
it=it+1; Jold=J;
q=-inv(L)*g; // (solve state equation)
g2=g'*g; q2=q'*q;
J=g2/q2; // (objective function)
dJrel=abs((J-Jold)/J);
a=-2*(inv(L')*q)*g2/q2^2; // (solve adjoint equation)
g=a*q2/2.0; // (enforce optimality eq.)
// Normalize g (optional)
//g=g/norm(g);
end // (end of iteration loop)
// optimal amplification
R=1.0/J;
// print results
it, R // (final iteration and amplification)
g // (optimal forcing)
q // (optimal response)
// normalize g and q
g=g/norm(g)
q=q/norm(q)
```

Optimal forcing: $\begin{pmatrix}1\\0\end{pmatrix}$; Optimal response: $\begin{pmatrix}0\\1\end{pmatrix}$; $R \longrightarrow 286221.23$

```
--> // print results

--> it, R // (final iteration and amplification)
 it  =

   4.
 R  =

   286221.23

--> g // (optimal forcing)
 g  =

   12.870384
   0.3199609

--> q // (optimal response)
 q  =

   514.81534
   6868.4707

--> // normalize g and q

--> g=g/norm(g)
 g  =

   0.9996911
   0.0248526

--> q=q/norm(q)
 q  =

   0.0747438
   0.9972028

-->
```

Optimal forcing: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$; Optimal response: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $R \longrightarrow 286221.23$

Step #9: Example: Optimal energy growth

Introduction of time and of a non linear model. We consider two constraints:

$$F(q) = \frac{dq}{dt} - N(q) = 0 \qquad \text{State equation}$$
$$F_0(q,g) = q(0) - g = 0 \qquad \text{Initial condition}$$

The control parameter is $g = q(0)$.

We search to maximize the temporal energy amplification:

$$G(T) = \frac{q(T)\cdot q(T)}{q(0)\cdot q(0)}$$

This is the ratio of the energy at time T with the energy at time 0. Maximizing G is equivalent to minimizing:

$$\mathcal{J}(q,g) = \frac{g\cdot g}{q(T)\cdot q(T)} = \left.\frac{g\cdot g}{q(t)\cdot q(t)}\right|_{t=T}$$

We have two constraints:

1. $\dfrac{dq}{dt} - N(q) = 0$

2. $q(0) - g = 0$

We follow the variational approach and modify the inner product since one of the constraint is depending on time. We introduce the following Lagrangian:

$$\mathcal{L}(q, g, a, b) = \mathcal{J}(q, g) - \int_0^T a(t) \underbrace{\left[\frac{dq}{dt} - N(q)\right]}_{F(q)} \mathrm{d}t - b \underbrace{[q(0) - g]}_{F_0(q,g)}$$

▷ **State equation**: variation with respect to the co-state a

$$\frac{\partial \mathcal{L}}{\partial a}\tilde{a} = \lim_{\epsilon \longrightarrow 0} \frac{\mathcal{L}(q, g, a + \epsilon\tilde{a}, b) - \mathcal{L}(q, g, a, b)}{\epsilon} = 0 \qquad \forall \tilde{a}$$

$$\Longrightarrow \int_0^T \tilde{a}(t) \left[\frac{dq}{dt} - N(q)\right] \mathrm{d}t = 0 \qquad \forall \tilde{a}$$

$$\Longrightarrow \frac{dq}{dt} = N(q)$$

▷ **State equation**: variation with respect to the co-state b

$$\frac{\partial \mathcal{L}}{\partial b}\tilde{b} = \lim_{\epsilon \longrightarrow 0} \frac{\mathcal{L}(q,g,a,b+\epsilon\tilde{b}) - \mathcal{L}(q,g,a,b)}{\epsilon} = 0 \qquad \forall \tilde{b}$$

$$\Longrightarrow \tilde{b}\,[q(0) - g] = 0 \qquad \forall \tilde{b}$$

$$\Longrightarrow q(0) = g$$

▷ **Adjoint equation**: variation with respect to the state q

$$\frac{\partial \mathcal{L}}{\partial q}\tilde{q} = \lim_{\epsilon \longrightarrow 0} \frac{\mathcal{L}(q+\epsilon\tilde{q}, g, a, b) - \mathcal{L}(q, g, a, b)}{\epsilon} = 0 \qquad \forall \tilde{q}$$

$$\begin{aligned}\mathcal{L}(q+\epsilon\tilde{q}, g, a, b) - \mathcal{L}(q, g, a, b) = {} & \mathcal{J}(q+\epsilon\tilde{q}, g) - \mathcal{J}(q, g) \\ & - \int_0^T a(t)\, \left[F(q+\epsilon\tilde{q}) - F(q)\right] \mathrm{d}t \\ & - b\, \left[F_0(q+\epsilon\tilde{q}, g) - F_0(q, g)\right]\end{aligned}$$

We perform Taylor developments to order 2, for $\mathcal{J}$, F and F_0.

$$\mathcal{J}(q+\epsilon\tilde{q}, g) = \mathcal{J}(q, g) + \epsilon \frac{\partial \mathcal{J}}{\partial q} \cdot \tilde{q} + \mathcal{O}(\epsilon^2)$$

$$\frac{\partial \mathcal{L}}{\partial q}\tilde{q} = \left.\frac{\partial \mathcal{J}}{\partial q(t)}\tilde{q}(t)\right|_{t=T} - \int_0^T a(t)\, \left(\frac{\partial F}{\partial q}\right) \tilde{q}\,\mathrm{d}t - b\, \left(\frac{\partial F_0}{\partial q}\right) \tilde{q} = 0 \qquad \forall \tilde{q}$$

$$\frac{\partial \mathcal{L}}{\partial q}\tilde{q} = \frac{\partial \mathcal{J}}{\partial q(t)}\tilde{q}(t)\bigg|_{t=T} - \int_0^T a(t)\,\left(\frac{\partial F}{\partial q}\right)\tilde{q}\,\mathrm{d}t - b\left(\frac{\partial F_0}{\partial q}\right)\tilde{q} = 0 \qquad \forall\tilde{q}$$

From the definition of F and F_0, we deduce:

$$\frac{\partial F}{\partial q} = \frac{dI}{dt} - \frac{\partial N}{\partial q} \quad \text{and} \quad \frac{\partial F_0}{\partial q} = I(0)$$

where I is the identity operator (for instance, $Iq = q$). We deduce:

$$\frac{\partial \mathcal{L}}{\partial q}\tilde{q} = \frac{\partial \mathcal{J}}{\partial q(T)}\tilde{q}(T) - \int_0^T a(t)\,\left(\frac{d\tilde{q}}{dt} - \frac{\partial N}{\partial q}\tilde{q}\right)\mathrm{d}t - b\,\tilde{q}(0) = 0 \qquad \forall\tilde{q} \qquad (1.1)$$

We now transform this expression with the goal to exploit the fact that this equations is verified $\forall\tilde{q}$. For that, we try to put $\tilde{q}$ in factor.

1. Since $\mathbf{u} \cdot A\mathbf{v} = A^T\mathbf{u} \cdot \mathbf{v}$, we can write:

$$\int_0^T a(t)\,\frac{\partial N}{\partial q}\tilde{q}\,\mathrm{d}t = \int_0^T \left(\frac{\partial N}{\partial q}\right)^T a \cdot \tilde{q}\,\mathrm{d}t$$

2. We integrate by parts:

$$\int_0^T a(t)\,\frac{d\tilde{q}}{dt}\,\mathrm{d}t = a(T)\,\tilde{q}(T) - a(0)\,\tilde{q}(0) - \int_0^T \tilde{q}\,\frac{da(t)}{dt}\,\mathrm{d}t$$

$$\Longrightarrow \frac{\partial \mathcal{J}}{\partial q(T)}\tilde{q}(T) - \left(a(T)\,\tilde{q}(T) - a(0)\,\tilde{q}(0)\right) - \int_0^T \left[-\frac{da(t)}{dt} - \left(\frac{\partial N}{\partial q}\right)^T a\right] \cdot \tilde{q}\,\mathrm{d}t - b\,\tilde{q}(0) = 0 \quad \forall \tilde{q}$$

From the definition of $\mathcal{J}$, we deduce:

$$\frac{\partial \mathcal{J}}{\partial q(T)} = -2q(T)\frac{g \cdot g}{\left(q(T) \cdot q(T)\right)^2} \quad \Longrightarrow$$

$$\tilde{q}(T)\left[-a(T) - 2q(T)\frac{g \cdot g}{\left(q(T) \cdot q(T)\right)^2}\right]$$
$$+\tilde{q}(0)\,\left(a(0) - b\right)$$
$$-\int_0^T \left[-\frac{da(t)}{dt} - \left(\frac{\partial N}{\partial q}\right)^T a\right] \cdot \tilde{q}\,\mathrm{d}t$$
$$= 0 \quad \forall \tilde{q}$$

We deduce:

1. Adjoint equation. $\frac{da(t)}{dt} = -\left(\frac{\partial N}{\partial q}\right)^T a \quad \text{for} \quad t \in [0;T]$

2. Adjoint equation. Terminal condition. $a(T) = -2q(T)\frac{g \cdot g}{(q(T) \cdot q(T))^2}$

3. Compatibility condition. $a(0) = b$.

▷ **Optimality condition**: variation with respect to the control g

$$\frac{\partial \mathcal{L}}{\partial g}\tilde{g} = \lim_{\epsilon \longrightarrow 0} \frac{\mathcal{L}(q, g+\epsilon\tilde{g}, a, b) - \mathcal{L}(q, g, a, b)}{\epsilon} = 0 \qquad \forall \tilde{g}$$

Reminder: $$\mathcal{L}(q,g,a,b) = \mathcal{J}(q,g) - \int_0^T a(t) \underbrace{\left[\frac{dq}{dt} - N(q)\right]}_{F(q)} \mathrm{d}t - b \underbrace{[q(0) - g]}_{F_0(q,g)}$$

$$\frac{\partial \mathcal{L}}{\partial g}\tilde{g} = \frac{\partial \mathcal{J}}{\partial g}\tilde{g} - b \underbrace{\left(\frac{\partial F_0}{\partial g}\right)}_{-I} \tilde{g} \; = 0 \qquad \forall \tilde{g}$$

$$= \left(\frac{\partial \mathcal{J}}{\partial g} + b\right)\tilde{g} = 0 \qquad \forall \tilde{g} \qquad \Longrightarrow \frac{\partial \mathcal{J}}{\partial g} + b = 0$$

From the definition of $\mathcal{J}$, we deduce:

$$\frac{\partial \mathcal{J}}{\partial g} + b = 0$$

$$\frac{\partial \mathcal{J}}{\partial g} = \frac{2g}{q(T)\cdot q(T)} \quad \Longrightarrow g = -b\frac{q(T)\cdot q(T)}{2}$$

We can use the compatibility equation from the adjoint equation to remove b in the optimal system. We get the following optimality system:

$$\frac{dq}{dt} = N(q) \quad ; \quad q(0) = g$$

$$-\frac{da(t)}{dt} = \left(\frac{\partial N}{\partial q}\right)^T a \quad \text{with} \quad a(T) = -2q(T)\frac{g\cdot g}{(q(T)\cdot q(T))^2}$$

$$g = -a(0)\frac{q(T)\cdot q(T)}{2}$$

Only one adjoint variable !!

Ex: $\dfrac{dq}{dt} = Lq$ with $L = \begin{bmatrix} -\dfrac{1}{\mathrm{Re}} & 0 \\ 1 & -\dfrac{3}{\mathrm{Re}} \end{bmatrix}$ *i.e.* $N(q) \triangleq Lq$, linear equation

for numerical facilities, $\Longrightarrow \left(\dfrac{\partial N}{\partial q}\right)^T = L^T$.

Solution of $\dfrac{dq}{dt} = Lq$ is given by:

$$q(t) = \exp(Lt)\, q(0) \quad \Longrightarrow$$

$$G(T) = \frac{q(T) \cdot q(T)}{q(0) \cdot q(0)} = \frac{\left\| \exp(LT)\, q(0) \right\|^2}{\left\| q(0) \right\|^2} \quad \Longrightarrow$$

$$\max_{q(0) \neq 0} G(T) = \max_{q(0) \neq 0} \frac{\left\| \exp(LT)\, q(0) \right\|^2}{\left\| q(0) \right\|^2} \triangleq \left\| \exp(LT) \right\|^2$$

Scilab program: Find $G_{\text{Max}} = \max_T G(T)$ for a given value of Re going from 1 to 1000. Comment on the non normality.

The optimality system now depends on time. We have to integrate forward in time the state equation and backward in time the adjoint equation. The system is solved iteratively. We consider a guess solution for the optimal forcing (g).

1. Solve the State equation

$$\frac{dq}{dt} = Lq \quad ; \quad q(0) = g$$

2. Solve the Adjoint equation

$$-\frac{da(t)}{dt} = L^T a \quad \text{with} \quad a(T) = -2q(T)\frac{g \cdot g}{(q(T) \cdot q(T))^2}$$

3. Solve the Optimality condition

$$g = -a(0)\frac{q(T) \cdot q(T)}{2}$$

$$\mathcal{J}(q, g) = \frac{g \cdot g}{q(T) \cdot q(T)} = \left.\frac{g \cdot g}{q(t) \cdot q(t)}\right|_{t=T}$$

Optimal_Temporal_Energy_Growth.sce

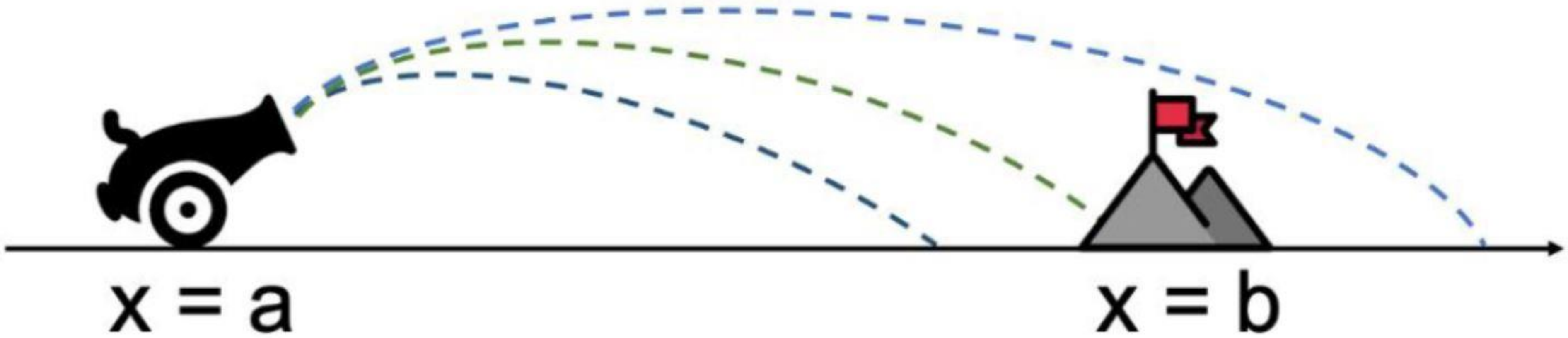
```
//**********************************;
//* F U N C T I O N S *
//**********************************;
// State equation rhs
function [f]=StateForw(t,q,L);
f=L*q
endfunction;
//**********************************;
// Adjoint equation rhs
function [f]=AdjntBack(t,a,L);
f=-L'*a
endfunction;
```

```
//*****************************;
//* M A I N P R O G R A M *
//*****************************;
// Define system
Rey=400.0
T=200.0
L=[-1.0/Rey 0; 1 -3.0/Rey]
// Exact solution using the norm function
G_exact=(norm(expm(L*T)))^2;
// Define tolerance and initialize iterations
tol=10^(-8);
g=[rand(); rand()]; // (random initial guess)
g=g/sqrt(g'*g); // normalize
norm(g); // Check the norm of the initial condition
J=10^23; dJrel=10^23; it=0;
```

```
// Iteration loop
while (dJrel>tol)
it=it+1; Jold=J;
// forward integration of evolution eq
q0=g; [qT]=ode(q0,0,T, list(StateForw, L));
g2=g'*g; qT2=qT'*qT;
J=g2/qT2;
dJrel=abs((J-Jold)/J);
// backward integration of adjoint eq
aT=-2*qT*(g2)/qT2^2;
[a0]=ode(aT,T,0,list(AdjntBack,L));
g=-a0*(qT2/2.0);// enforce the optimality equation
end
// end of iteration loop
// print results
G=1.0/J;
it, G, G_exact
g // optimal initial condition
g=g/sqrt(g'*g) // normalize
qT // optimal response
// end of program
```

```
--> // print results

--> G=1.0/J;

--> it, G, G_exact
 it  =


   3.
 G  =


   5880.2559
 G_exact  =


   5880.2554

--> g // optimal initial condition
 g  =


   1.2819809
   0.0037302

--> g=g/sqrt(g'*g) // normalize
 g  =


   0.9999958
   0.0029097

--> qT // optimal response
 qT  =


   0.7775628
   98.30352

--> // end of program
```

Step #10: Example: Final target state with time-dependent forcing



Example of shooting method.

The control function $g = g(t)$ is now time dependent. We consider:

$$\frac{dq}{dt} = f(q, g) \quad \text{with } g = g(t)$$
$$q(0) = q_0$$

Determine $g(t)$ such that the final state targets to P, *i.e.* $q(T) \longrightarrow P$. We introduce:

$$\mathcal{J} = \underbrace{\frac{1}{2}\left(q(T) - P\right)\left(q(T) - P\right)}_{\text{Performance term}} + \underbrace{\frac{\gamma^2}{2}\overbrace{\int_0^T (g \cdot g)\, \mathrm{d}t}^{\text{Energy control}}}_{\text{Penalization term}}$$

We consider the Lagrangian formalism and introduce:

$$\mathcal{L}(q, g, a, b) = \mathcal{J}(q, g) - \int_0^T a(t) \left[\frac{dq}{dt} - f(q, g)\right] \mathrm{d}t - b\,[q(0) - q_0]$$

We derive the optimality system.

▷ **Adjoint equation**: variation with respect to the state q

$$\lim_{\epsilon \longrightarrow 0} \frac{\mathcal{L}(q + \epsilon\tilde{q}, g, a, b) - \mathcal{L}(q, g, a, b)}{\epsilon} = 0 \qquad \forall \tilde{q}$$

By modifying the notations used in (1.1), we obtain:

$$\frac{\partial \mathcal{J}}{\partial q(T)}\tilde{q}(T) - \int_0^T a(t) \left(\frac{d\tilde{q}}{dt} - \frac{\partial f}{\partial q}\tilde{q}\right) \mathrm{d}t - b\,\tilde{q}(0) = 0 \qquad \forall \tilde{q}$$

$$\frac{\partial \mathcal{J}}{\partial q(T)}\tilde{q}(T) - \int_0^T a(t)\left(\frac{d\tilde{q}}{dt} - \frac{\partial f}{\partial q}\tilde{q}\right)\mathrm{d}t - b\,\tilde{q}(0) = 0 \qquad \forall \tilde{q}$$

Similarly as before, we modify this expression to introduce $\tilde{q}$ in factor. For that, we use the same two tricks:

1.
$$\int_0^T a(t)\,\frac{\partial f}{\partial q}\tilde{q}\,\mathrm{d}t = \int_0^T \left(\frac{\partial f}{\partial q}\right)^T a \cdot \tilde{q}\,\mathrm{d}t$$

2. Integration by parts:

$$\int_0^T a(t)\,\frac{d\tilde{q}}{dt}\,\mathrm{d}t = a(T)\,\tilde{q}(T) - a(0)\,\tilde{q}(0) - \int_0^T \tilde{q}\,\frac{da(t)}{dt}\,\mathrm{d}t$$

$$\Longrightarrow \frac{\partial \mathcal{J}}{\partial q(T)}\tilde{q}(T) - (a(T)\,\tilde{q}(T) - a(0)\,\tilde{q}(0)) - \int_0^T \left[-\frac{da(t)}{dt} - \left(\frac{\partial f}{\partial q}\right)^T a\right] \cdot \tilde{q}\,\mathrm{d}t - b\,\tilde{q}(0) = 0 \quad \forall \tilde{q}$$

From the definition of $\mathcal{J}$, we deduce:

$$\frac{\partial \mathcal{J}}{\partial q(T)} = q(T) - P \quad \Longrightarrow$$

$$\begin{aligned} &\tilde{q}(T)\,\left[-a(T) + q(T) - P\right] \\ &+\tilde{q}(0)\,\left(a(0) - b\right) \\ &- \int_0^T \left[-\frac{da(t)}{dt} - \left(\frac{\partial f}{\partial q}\right)^T a\right] \cdot \tilde{q}\,\mathrm{d}t \\ &= 0 \quad \forall \tilde{q} \end{aligned}$$

$$
\begin{aligned}
&\tilde{q}(T)\ [-a(T) + q(T) - P] \\
&+\tilde{q}(0)\ (a(0) - b) \\
&- \int_0^T \left[-\frac{da(t)}{dt} - \left(\frac{\partial f}{\partial q} \right)^T a \right] \cdot \tilde{q}\, \mathrm{dt} \\
&= 0 \quad \forall \tilde{q}
\end{aligned}
$$

We deduce:

1. Adjoint equation. $-\dfrac{da(t)}{dt} = \left(\dfrac{\partial f}{\partial q} \right)^T a(t)$

2. Adjoint equation. Terminal condition. $a(T) = q(T) - P$

3. Compatibility condition. $a(0) = b$.

▷ **Optimality condition**: variation with respect to the control g

$$\lim_{\epsilon \longrightarrow 0} \frac{\mathcal{L}(q, g+\epsilon\tilde{g}, a, b) - \mathcal{L}(q, g, a, b)}{\epsilon} = 0 \qquad \forall \tilde{g}$$

<u>Reminder</u>: $\mathcal{L}(q, g, a, b) = \mathcal{J}(q, g) - \int_0^T a(t) \left[\frac{dq}{dt} - f(q, g)\right] \mathrm{d}t - b\,[q(0) - q_0]$

We obtain (almost immediately) that:

$$\frac{\partial \mathcal{J}}{\partial g}\tilde{g} - \int_0^T a(t) \left(-\frac{\partial f}{\partial g}\tilde{g}\right) \mathrm{d}t = 0 \qquad \forall \tilde{g}$$

$$\frac{\partial \mathcal{J}}{\partial g}\tilde{g} + \int_0^T \left(\frac{\partial f}{\partial g}\right)^T a \cdot \tilde{g}\,\mathrm{d}t = 0 \qquad \forall \tilde{g}$$

$$\frac{\partial \mathcal{J}}{\partial g}\tilde{g} + \int_0^T \left(\frac{\partial f}{\partial g}\right)^T a \cdot \tilde{g}\,\mathrm{d}t = 0 \qquad \forall \tilde{g}$$

Since $\mathcal{J} = \frac{1}{2}\left(q(T) - P\right)\left(q(T) - P\right) + \frac{\gamma^2}{2}\int_0^T (g \cdot g)\,\mathrm{d}t$, we got

$$\frac{\partial \mathcal{J}}{\partial g}\tilde{g} = \gamma^2 \int_0^T g \cdot \bullet\,\mathrm{d}t \quad \text{operator notation} \quad \Longrightarrow$$

$$\gamma^2 \int_0^T g \cdot \tilde{g}\,\mathrm{d}t \quad + \int_0^T \left(\frac{\partial f}{\partial g}\right)^T a \cdot \tilde{g}\,\mathrm{d}t = 0 \qquad \forall \tilde{g}$$

$$\int_0^T \left(\gamma^2 g + \left(\frac{\partial f}{\partial g}\right)^T a\right) \cdot \tilde{g}\,\mathrm{d}t = 0 \qquad \forall \tilde{g}$$

$$\Longrightarrow g = -\frac{1}{\gamma^2}\left(\frac{\partial f}{\partial g}\right)^T a \quad \text{Optimality condition}$$

We get the following optimality system:

$$\frac{dq}{dt} = f(q,g) \quad ; \quad q(0) = q_0 \quad \text{State equation}$$

$$-\frac{da(t)}{dt} = \left(\frac{\partial f}{\partial q}\right)^T a \quad \text{with} \quad a(T) = q(T) - P \quad \text{Adjoint equation}$$

$$g = -\frac{1}{\gamma^2}\left(\frac{\partial f}{\partial g}\right)^T a \quad \text{Optimality condition}$$

Step #11: Example: Optimal temporal distribution of heating

Objective: Target the temperature $\theta(t)$ at time T to P.
We introduce the state variable q as

$$q(t) = \theta(t) - \theta_e$$

where θ_e is the external temperature. We assume that $\theta(0) = \theta_e$, *i.e.* $q(0) = 0$. The state equation is given by

$$\begin{aligned} \frac{dq}{dt} &= -Aq + Bg \quad \text{with } g : \text{heating depending on time} \\ q(0) &= q_0 \end{aligned}$$

The cost function is:

$$\mathcal{J} = \frac{1}{2}\left(q(T) - P\right)^2 + \frac{\gamma^2}{2}\int_0^T g^2 \, \mathrm{d}t$$

We are exactly in the framework of Step #10 with

$$f = -Aq + Bg \quad \Longrightarrow \quad \frac{\partial f}{\partial q} = -A \quad \text{and} \quad \frac{\partial f}{\partial g} = B$$

Optimality system:

$$\frac{dq}{dt} = -Aq + Bg \quad ; \quad q(0) = q_0 \quad \text{State equation}$$

$$\frac{da(t)}{dt} = A\,a(t) \quad \text{with} \quad a(T) = q(T) - P \quad \text{Adjoint equation}$$

$$g = -\frac{1}{\gamma^2} B\,a \quad \text{Optimality condition}$$

Analytical solution: We insert the expression of g in the equation for q. We got:

$$\frac{d\Phi}{dt} = C\Phi \quad \Longrightarrow \quad \boxed{\Phi(t) = \exp\left(Ct\right)\Phi(0)} \tag{1.2}$$

where

$$\Phi = \begin{pmatrix} q \\ a \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} -A & -\dfrac{1}{\gamma^2}B^2 \\ 0 & A \end{pmatrix}$$

We note

$$M(t) = Ct = \begin{pmatrix} -At & -\dfrac{1}{\gamma^2}B^2t \\ 0 & At \end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ 0 & c(t) \end{pmatrix}$$

By definition of the exponent of a matrix, we have:

$$\exp\left(Ct\right) = \exp\left(M(t)\right) = \sum_{k\in\mathbb{N}} \frac{1}{k!} M^k$$

We can prove that

$$M^k = \begin{pmatrix} a^k(t) & \alpha_k(t) \\ 0 & c^k(t) \end{pmatrix}$$

where

$$\alpha_k = b\left(a^{k-1}c^0 + a^{k-2}c + \cdots + a^0c^{k-1}\right) = b\frac{a^k - c^k}{a - c}$$

We finally deduce that:

$$\exp\left(M(t)\right) = \begin{pmatrix} \exp\left(a(t)\right) & x(t) \\ 0 & \exp\left(c(t)\right) \end{pmatrix}$$

where

$$x(t) = \frac{b\left(\exp\left(a(t)\right) - \exp\left(c(t)\right)\right)}{a - c} = -\frac{1}{\gamma^2}\frac{B^2}{A}\sinh\left(At\right)$$

The solution of (1.2) is given by:

$$\Phi(t) = \begin{pmatrix} q(t) \\ a(t) \end{pmatrix} = \begin{pmatrix} \exp\left(-A(t)\right) & x(t) \\ 0 & \exp\left(A(t)\right) \end{pmatrix} \begin{pmatrix} q(0) \\ a(0) \end{pmatrix} \quad \text{i.e.}$$

$$\boxed{q(t) = \exp\left(-A(t)\right) q(0) + x(t)a(0)} \quad \text{and} \quad \boxed{a(t) = \exp\left(A(t)\right) a(0)}$$

Using the terminal condition $a(T) = q(T) - P$ and assuming that $q(0) = 0$, we can prove that:

$$q(t) \triangleq q_{\text{opt}}(t) = P\,\frac{B^2}{\gamma^2}\,\frac{\sinh(At)}{A\exp(AT) + \dfrac{B^2}{\gamma^2}\sinh(AT)}$$

and

$$g(t) \triangleq g_{\text{opt}}(t) = P\,\frac{AB}{\gamma^2}\,\frac{\exp(At)}{A\exp(AT) + \dfrac{B^2}{\gamma^2}\sinh(AT)}$$

Scilab program: Solve the problem for $A = B = 1$; $P = 1$; $T = 6$.

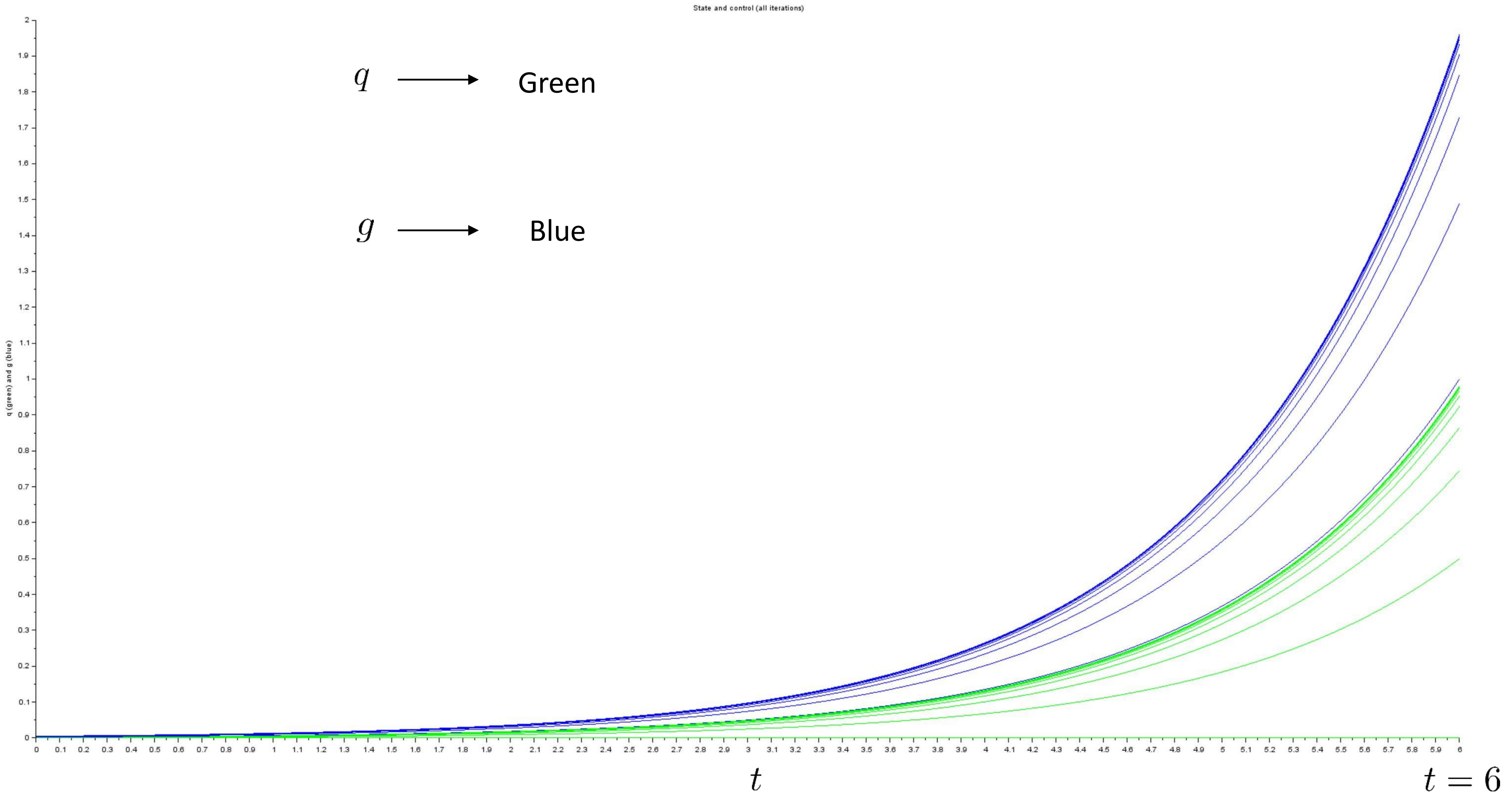
Drive_to_Final_Target_State_by_Time_Dependent_Forcing.sce

```
//*******************************;
//* F U N C T I O N S *
//*******************************;
// Returns the rhs state equation;
function [rhs]=StaEqnRHS(tloc, q, g, A, B);
rhs=-A*q+B*g;
endfunction;
//*******************************;
// Returns the rhs of the adjoint;
function [rhs]=AdjEqnRHS(tloc, a, A);
rhs=A*a
endfunction;
//*******************************;
//* M A I N P R O G R A M *
//*******************************;
// Here we give a value to the parameters
// and choose the initial conditions
A=1.0, B=1.0 // coefficients of the equation
q0=0.0 // initial condition
p=1.0 // target temperature
T=6.0 // target time
gam2=0.01 // weight of control cost gamma^2
maxiter=15 // iterations
Nt=200 // retained time samples
alpha=min(0.5,gam2) // relaxation factor
for j=1:Nt;
    t(j)=(j - 1)*T/(Nt - 1); // time grid
end;
g=0.0*t; //initialize to zero the control;
// perform iterations up to maxiter
```

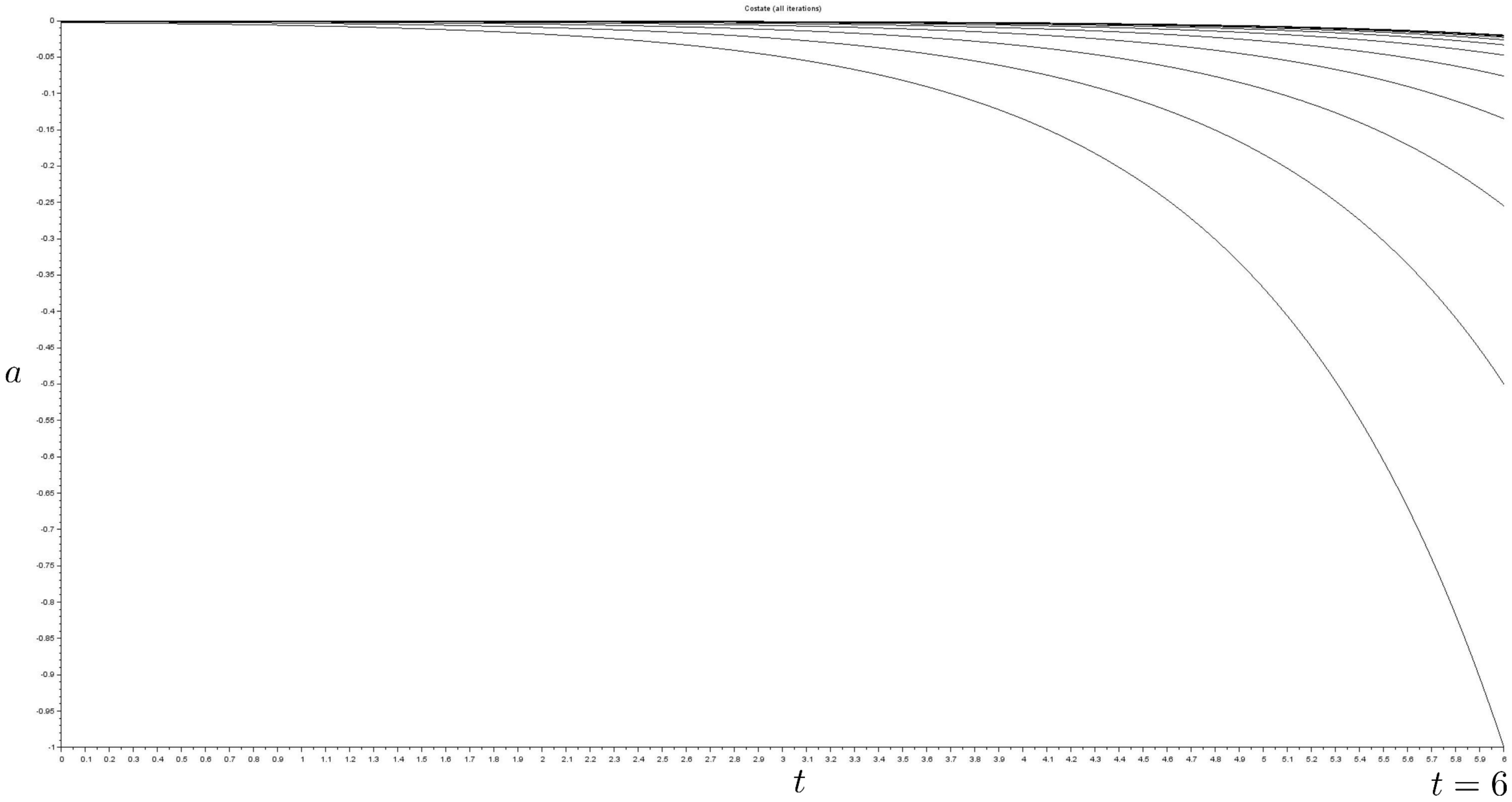
```
// perform iterations up to maxiter
for iter=1:maxiter
// Integrate state eqn forward in time;
    q(1)=q0; // give IC;
    for j=2:Nt;
// local time interval and local IC
        t_i=t(j - 1); t_f=t(j); q_i=q(j - 1);
// local control in the time interval:
        gloc=0.5*(g(j)+g(j - 1));
// integrate forward and store solution in q_f
        [q_f]=ode(q_i,t_i,t_f,list(StaEqnRHS, gloc, A, B));
        q(j)=q_f;
    end
// compute cost of control and total cost
    g2int=0.5*g(1)^2+sum(g(2:Nt - 1)^2)+0.5*g(Nt)^2;
    Jg(iter)=0.5*gam2*(T/(Nt - 1))*g2int;
    J(iter)=0.5*(q(Nt) - p)^2+Jg(iter);
// Integrate the adjoint equations backward in time
// enforce IC (at T) for backward integration
    a(Nt)=(q(Nt) - p);
    for j=Nt-1:-1:1;
// local time interval
        t_i=t(j+1); t_f=t(j); a_i=a(j+1);
// integrate backward and store solution in a_f
        [a_f] =ode(a_i,t_i,t_f,list(AdjEqnRHS, A));
        a(j)=a_f;
    end
// Enforce optimality cond. using under-relaxation;
    g=(1 - alpha)*g+alpha*(-B/gam2)*a;
// and plot the result of the current iteration;
// Plot q(t) and g(t);
    xset("window",0);
    xtitle('State and control (all iterations)', 't', 'q (green) and g (blue
    plot2d(t',q',style=3); // style = 3 green
    plot2d(t',g',style=2); // style = 2 blue
// Plot a(t);
    xset("window",1); xtitle("Costate (all iterations)", 't', 'a');
    plot2d(t', a',style=1);
end // end of the iteration loop
pause
```

```
pause
// Compare to exact solutions computed for q0=0;
DEN=(A*exp(A*T)+(B^2/gam2)*sinh(A*T));
q_ex=p*(B^2/gam2)*sinh(A*t)/DEN;
g_ex=p*(A*B/gam2)*exp(A*t)/DEN;
xset("window", 3); clf();
xtitle('q versus q_ex', 't', 'q');
plot2d(t',q_ex', style=-3); plot2d(t',q', style=3);
xset("window",4); clf();
xtitle('g versus g_ex', 't', 'g');
plot2d(t',g_ex',style=-2); plot2d(t', g', style=2);
// Print cost function history and convergence
for iter=2:maxiter
dJrel(iter)=abs(1.0 - J(iter - 1)/J(iter));
end
[[1:maxiter]' J dJrel]
```

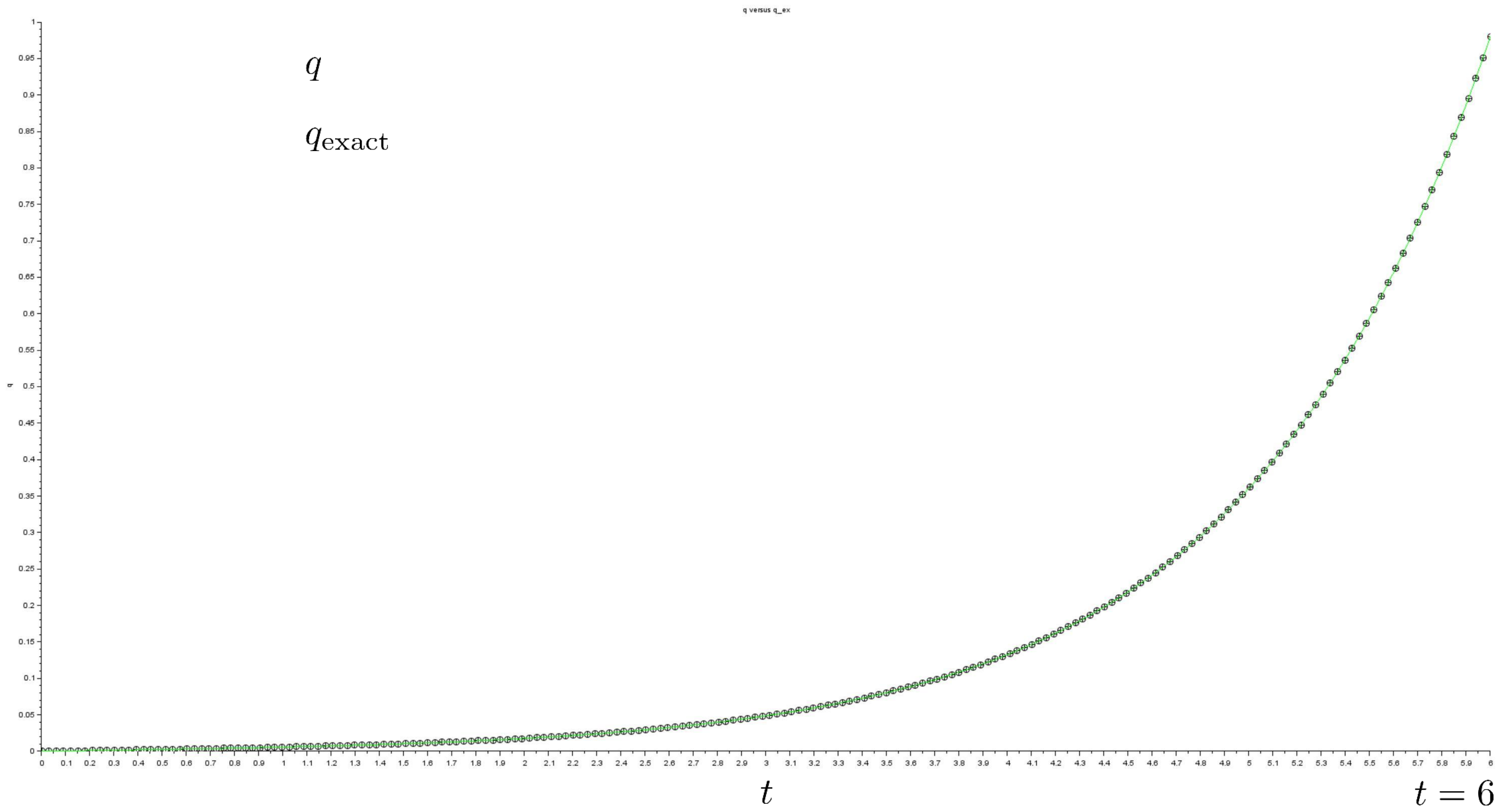
Time evolution of state and control variables at all iterations.



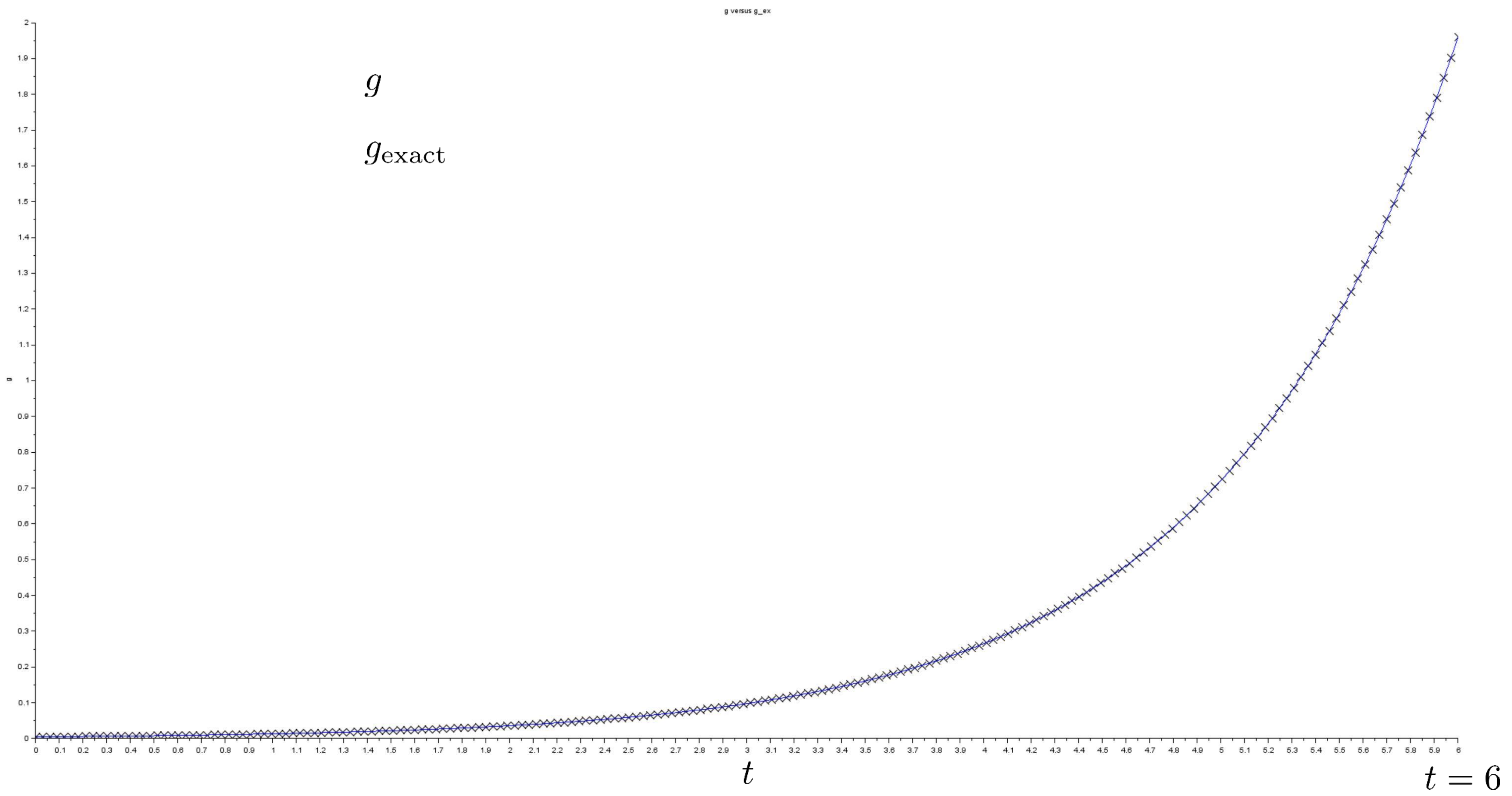
Time evolution of costate variables at all iterations.



Comparison of q and q_{exact}.



Comparison of g and g_{exact}.



Block diagrams for feedback control

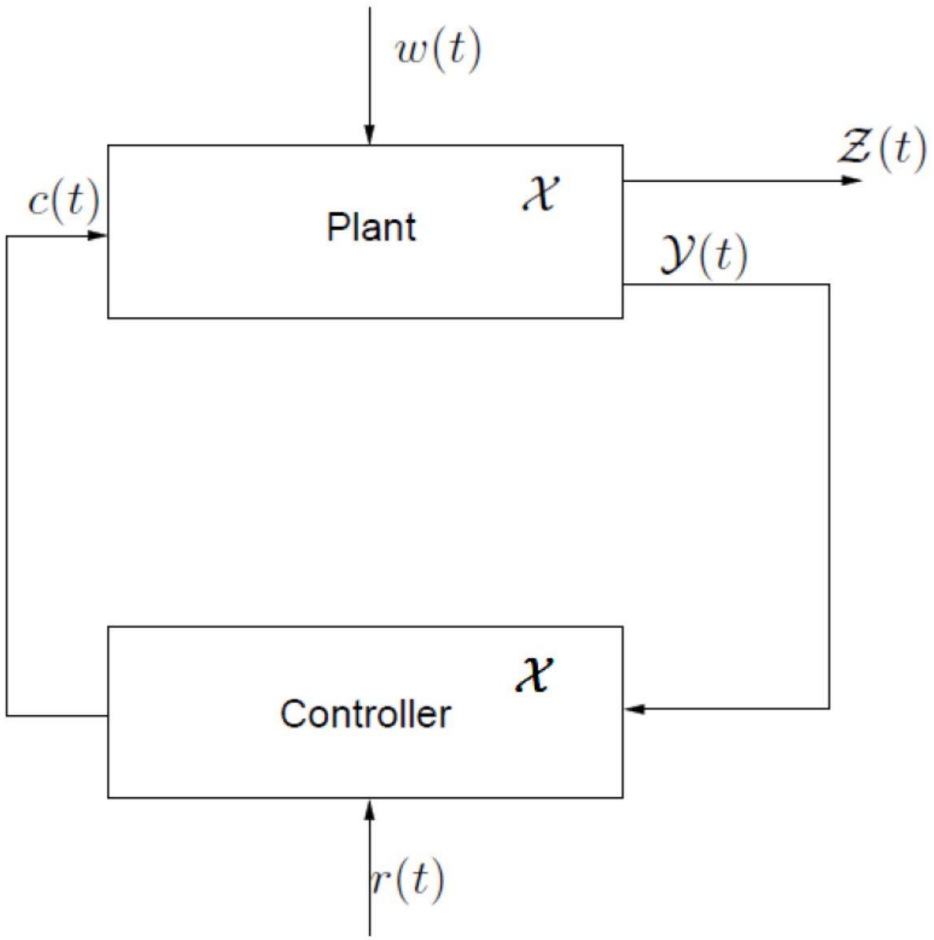


Fig. : *Full state configuration.*

Terminology:

- Plant :

1. Model-based approaches
   - White-box model
   - Grey-box model
   - Black-box model
2. Model-free approaches

- $\mathcal{X}$: states
- $\mathcal{Y}$: measured output or observable
- $\mathcal{Z}$: reference output
- c: input or control
- w: disturbance input
- r: reference input

▷ Linear state-space model

$$\boldsymbol{F}(\boldsymbol{x},\boldsymbol{u}) = \dot{\boldsymbol{x}} - A\,\boldsymbol{x} - B_2\boldsymbol{u} = 0$$

▷ Cost functional

$$\begin{aligned}\mathcal{J} &= \frac{1}{2}\int_0^T \left(\boldsymbol{x}^H\, C_1^H C_1\, \boldsymbol{x} + \ell^2\, \boldsymbol{u}^H\, \boldsymbol{u}\right)\,\mathrm{d}t \\ &= \frac{1}{2}\left[\langle C_1\boldsymbol{x}, C_1\boldsymbol{x}\rangle + \ell^2\,\langle \boldsymbol{u},\boldsymbol{u}\rangle\right]\end{aligned}$$

where $\langle \boldsymbol{a},\boldsymbol{b}\rangle = \int_0^T \boldsymbol{a}^H(t)\boldsymbol{b}(t)\,\mathrm{d}t +$ complex conjugate.

Determine the solution $\boldsymbol{x}$ and the control parameter $\boldsymbol{u}$ such that the cost functional $\mathcal{J}$ reaches a minimum.

▷ Lagrangian functional where $\boldsymbol{x}^+(t)$ is to enforce $\boldsymbol{F}$

$$\mathcal{L}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{x}^+) \triangleq \mathcal{J}(\boldsymbol{x},\boldsymbol{u}) - \left\langle \boldsymbol{F}(\boldsymbol{x},\boldsymbol{u}),\boldsymbol{x}^+\right\rangle$$

Determine the solution $\boldsymbol{x}$, the control parameter $\boldsymbol{u}$ and the Lagrange multipliers $\boldsymbol{x}^+$ such that the Lagrangian functional $\mathcal{L}$ reaches a minimum.

▷ Direct problem

$$
\begin{aligned}
\left\langle \boldsymbol{\nabla}_{\boldsymbol{x}^+}\mathcal{L}, \delta\boldsymbol{x}^+\right\rangle &= \lim_{\epsilon\to 0}\frac{\mathcal{L}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{x}^+ + \epsilon\delta\boldsymbol{x}^+) - \mathcal{L}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{x}^+)}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{\mathcal{J}(\boldsymbol{x},\boldsymbol{u}) - \mathcal{J}(\boldsymbol{x},\boldsymbol{u})}{\epsilon} \\
&- \lim_{\epsilon\to 0}\frac{\langle \dot{\boldsymbol{x}} - A\,\boldsymbol{x} - B_2\boldsymbol{u}, \boldsymbol{x}^+ + \epsilon\delta\boldsymbol{x}^+\rangle - \langle \dot{\boldsymbol{x}} - A\,\boldsymbol{x} - B_2\boldsymbol{u}, \boldsymbol{x}^+\rangle}{\epsilon} \\
&= -\left\langle \dot{\boldsymbol{x}} - A\,\boldsymbol{x} - B_2\boldsymbol{u}, \delta\boldsymbol{x}^+\right\rangle = 0 \quad \forall\delta\boldsymbol{x}^+
\end{aligned}
$$

$\Longrightarrow$

$$\boxed{\dot{\boldsymbol{x}} = A\,\boldsymbol{x} + B_2\boldsymbol{u}}$$ State equations

▷ Adjoint problem

$$\begin{aligned}
\langle \nabla_{\boldsymbol{x}} \mathcal{L}, \delta \boldsymbol{x} \rangle &= \lim_{\epsilon \to 0} \frac{\mathcal{L}(\boldsymbol{x} + \epsilon \delta \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{x}^+) - \mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{x}^+)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\mathcal{J}(\boldsymbol{x} + \epsilon \delta \boldsymbol{x}, \boldsymbol{u}) - \mathcal{J}(\boldsymbol{x}, \boldsymbol{u})}{\epsilon} \\
&- \lim_{\epsilon \to 0} \frac{\left\langle (\boldsymbol{x} + \dot{\ } \epsilon \delta \boldsymbol{x}) - A\,(\boldsymbol{x} + \epsilon \delta \boldsymbol{x}) - B_2 \boldsymbol{u}, \boldsymbol{x}^+ \right\rangle - \langle \dot{\boldsymbol{x}} - A\,\boldsymbol{x} - B_2 \boldsymbol{u}, \boldsymbol{x}^+ \rangle}{\epsilon} \\
&= \underbrace{\frac{\partial \mathcal{J}}{\partial \boldsymbol{x}} \delta \boldsymbol{x}}_{T_I} - \underbrace{\left\langle (\dot{\delta \boldsymbol{x}}), \boldsymbol{x}^+ \right\rangle}_{T_{II}} + \underbrace{\left\langle A \delta \boldsymbol{x}, \boldsymbol{x}^+ \right\rangle}_{T_{III}}
\end{aligned}$$

Some transformations:

$$\begin{aligned}
\triangleleft\, T_I &= \lim_{\epsilon \to 0} \frac{\mathcal{J}(\boldsymbol{x} + \epsilon \delta \boldsymbol{x}, \boldsymbol{u}) - \mathcal{J}(\boldsymbol{x}, \boldsymbol{u})}{\epsilon} \\
&= \frac{1}{2} \left[\langle C_1 \delta \boldsymbol{x}, C_1 \boldsymbol{x} \rangle + \langle C_1 \boldsymbol{x}, C_1 \delta \boldsymbol{x} \rangle \right] = \langle C_1 \delta \boldsymbol{x}, C_1 \boldsymbol{x} \rangle = \left\langle \delta \boldsymbol{x}, C_1^+ C_1 \boldsymbol{x} \right\rangle
\end{aligned}$$

$$T_{II} = \left\langle (\dot{\delta \boldsymbol{x}}), \boldsymbol{x}^+ \right\rangle = \int_0^T \left(\dot{\delta \boldsymbol{x}}\right)^H \boldsymbol{x}^+ \, \mathrm{d}t + \text{c.c.} = \left[\delta \boldsymbol{x}^H \boldsymbol{x}^+\right]_0^T - \int_0^T \delta \boldsymbol{x}^H \dot{\boldsymbol{x}^+} \, \mathrm{d}t + \text{c.c.}$$

Since $\delta \boldsymbol{x}(0) = \mathbf{0}$ and considering that $\boldsymbol{x}^+(T) = \mathbf{0}$, we obtain: $T_{II} = \left\langle \delta \boldsymbol{x}, -\dot{\boldsymbol{x}^+} \right\rangle$.

◁

$$T_{III} = \langle A \delta \boldsymbol{x}, \boldsymbol{x}^+ \rangle = \langle \delta \boldsymbol{x}, A^+ \boldsymbol{x}^+ \rangle$$

$\Longrightarrow$

$$\begin{aligned} \langle \boldsymbol{\nabla}_{\boldsymbol{x}} \mathcal{L}, \delta \boldsymbol{x} \rangle &= \langle \delta \boldsymbol{x}, C_1^+ C_1 \boldsymbol{x} \rangle + \left\langle \delta \boldsymbol{x}, \dot{\boldsymbol{x}^+} \right\rangle + \langle \delta \boldsymbol{x}, A^+ \boldsymbol{x}^+ \rangle \\ &= \left\langle \delta \boldsymbol{x}, C_1^+ C_1 \boldsymbol{x} + \dot{\boldsymbol{x}^+} + A^+ \boldsymbol{x}^+ \right\rangle = 0 \quad \forall \delta \boldsymbol{x} \end{aligned}$$

$\Longrightarrow$

$$\boxed{-\dot{\boldsymbol{x}^+} = A^+ \boldsymbol{x}^+ + C_1^+ C_1 \boldsymbol{x}}$$ Adjoint equations

$$\boxed{\boldsymbol{x}^+(T) = \mathbf{0}}$$ T.C.

▷ Optimality conditions

$$
\begin{aligned}
\langle \boldsymbol{\nabla}_{\boldsymbol{u}}\mathcal{L}, \delta\boldsymbol{u}\rangle &= \lim_{\epsilon\to 0} \frac{\mathcal{L}(\boldsymbol{x}, \boldsymbol{u}+\epsilon\delta\boldsymbol{u}, \boldsymbol{x}^+) - \mathcal{L}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{x}^+)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{\mathcal{J}(\boldsymbol{x}, \boldsymbol{u}+\epsilon\delta\boldsymbol{u}) - \mathcal{J}(\boldsymbol{x}, \boldsymbol{u})}{\epsilon} \\
&- \lim_{\epsilon\to 0} \frac{\langle \dot{\boldsymbol{x}} - A\,\boldsymbol{x} - B_2(\boldsymbol{u}+\epsilon\delta\boldsymbol{u}), \boldsymbol{x}^+\rangle - \langle \dot{\boldsymbol{x}} - A\,\boldsymbol{x} - B_2\boldsymbol{u}, \boldsymbol{x}^+\rangle}{\epsilon} \\
&= \underbrace{\frac{\partial \mathcal{J}}{\partial \boldsymbol{u}}\delta\boldsymbol{u}}_{T_I} + \underbrace{\left\langle B_2\delta\boldsymbol{u}, \boldsymbol{x}^+\right\rangle}_{T_{II}}
\end{aligned}
$$

Some transformations:

$$\triangleleft\, T_I = \lim_{\epsilon\to 0} \frac{\mathcal{J}(\boldsymbol{x}, \boldsymbol{u}+\epsilon\delta\boldsymbol{u}) - \mathcal{J}(\boldsymbol{x}, \boldsymbol{u})}{\epsilon} = \frac{\ell^2}{2}\left[\langle \boldsymbol{u}, \delta\boldsymbol{u}\rangle + \langle \delta\boldsymbol{u}, \boldsymbol{u}\rangle\right] = \ell^2 \langle \delta\boldsymbol{u}, \boldsymbol{u}\rangle$$

$$\triangleleft\, T_{II} = \left\langle B_2\delta\boldsymbol{u}, \boldsymbol{x}^+\right\rangle = \left\langle \delta\boldsymbol{u}, B_2^+\boldsymbol{x}^+\right\rangle \qquad \Longrightarrow$$

$$\langle \boldsymbol{\nabla}_{\boldsymbol{u}}\mathcal{L}, \delta \boldsymbol{u}\rangle = \ell^2 \langle \delta \boldsymbol{u}, \boldsymbol{u}\rangle + \langle \delta \boldsymbol{u}, B_2^+ \boldsymbol{x}^+ \rangle = 0 \qquad \forall \delta \boldsymbol{u}$$

$\Longrightarrow$

$$\boxed{B_2^+ \boldsymbol{x}^+ = -\ell^2 \boldsymbol{u}} \qquad \text{Optimality conditions}$$

and

$$\boldsymbol{\nabla}_{\boldsymbol{u}}\mathcal{L} = B_2^+ \boldsymbol{x}^+ + \ell^2 \boldsymbol{u}$$

STATE EQUATION:

$$\boxed{\dot{\boldsymbol{x}} = A\,\boldsymbol{x} + B_2\boldsymbol{u}}$$

$$\boxed{\boldsymbol{x}(0) = \boldsymbol{x_0}} \qquad \text{(I.C.)}$$

COST FUNCTIONAL:

$$\boxed{\mathcal{J} = \frac{1}{2}\int_0^T \left(\boldsymbol{x}^H\, C_1^H C_1\, \boldsymbol{x} + \ell^2\, \boldsymbol{u}^H\, \boldsymbol{u}\right)\, \mathrm{d}t}$$

ADJOINT EQUATION:

$$\boxed{-\dot{\boldsymbol{x}}^+ = A^+\boldsymbol{x}^+ + C_1^+ C_1\boldsymbol{x}}$$

$$\boxed{\boldsymbol{x}^+(T) = \boldsymbol{0}} \qquad \text{(T.C.)}$$

OPTIMALITY CONDITION:

$$\boxed{B_2^+\boldsymbol{x}^+ = -\ell^2\boldsymbol{u}}$$

▷ Optimality system: three unknowns $\boldsymbol{x}$, $\boldsymbol{u}$ and $\boldsymbol{x}^+$.

▷ 1. Remove $\boldsymbol{u}$ from the optimality system

$$\text{optimality condition} \implies \boldsymbol{u}(t) = -\frac{1}{\ell^2} B_2^+ \boldsymbol{x}^+(t)$$

▷ 2. Insert this relation into the direct and adjoint equations. We obtain an Hamiltonian system given by

$$\begin{pmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{x}}^+ \end{pmatrix} = \begin{pmatrix} A & -\frac{1}{\ell^2} B_2 B_2^+ \\ -C_1^+ C_1 & -A^+ \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x}^+ \end{pmatrix} \quad \text{with} \quad \begin{cases} \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{x}^+(T) = \boldsymbol{0} \end{cases}$$

▷ 3. Consider that $\boldsymbol{x}^+(t) = \Pi(t)\boldsymbol{x}(t)$ where $\Pi \in \mathbb{R}^{n_x \times n_x}$

▷ 4. Obtain continuous time algebraic Riccati equation (CARE)

$$\boxed{-\dot{\Pi} = A^+ \Pi + \Pi A - \frac{1}{\ell^2} \Pi B_2 B_2^+ \Pi + C_1^+ C_1} \quad \text{with} \quad \Pi(T) = 0$$

▷ 5. Kalman gain

$$\boldsymbol{u}(t) = K(t)\boldsymbol{x}(t) \quad \text{with} \quad \boxed{K(t) = -\frac{1}{\ell^2} B_2^+ \Pi(t)}$$

Linearized channel flow

▷ The Navier-Stokes equations linearized about a base flow $\boldsymbol{U} = (U(y), 0, 0)$ write:

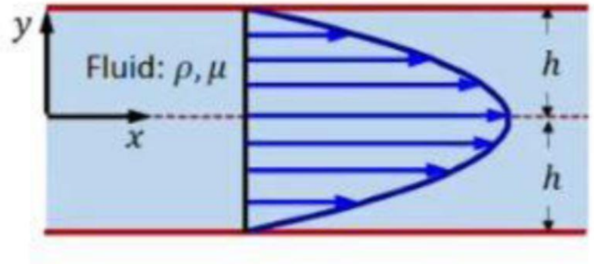
$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \qquad \mathbf{u} \text{ small perturbation around } \mathbf{U}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + U(y)\frac{\partial \boldsymbol{u}}{\partial x} + (U'(y)v, 0, 0) = -\boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot \left[\nu_T(y)\left(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u}^T\right)\right] + \boldsymbol{f}$$

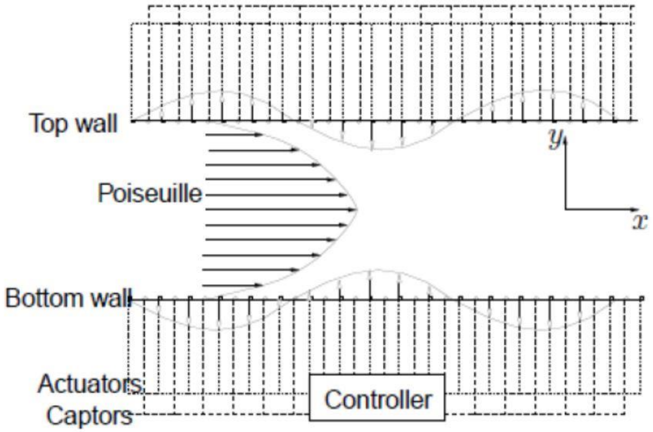
with $\nu_T = \frac{1}{Re}\left(1 + \frac{\nu_t}{\nu}\right)$ where ν_t is a turbulent viscosity.

▷ Mean velocity field

$$U(y) = \frac{dP_w}{dx}\int_{y_w}^{y} \frac{\eta}{\nu_T(\eta)}\,\mathrm{d}\eta$$



i.e. $U(y) = 1 - y^2$ for Poiseuille flow.



Linearized channel flow

$$\left(\frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x}\right) u + U'(y) v = -\frac{\partial p}{\partial x} + \nu_T(y)\Delta u + \nu_T'(y)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + f_x \quad (1)$$

$$\left(\frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x}\right) v = -\frac{\partial p}{\partial y} + \nu_T(y)\Delta v + 2\nu_T'(y)\frac{\partial v}{\partial y} + f_y \quad (2)$$

$$\left(\frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x}\right) w = -\frac{\partial p}{\partial z} + \nu_T(y)\Delta w + \nu_T'\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) + f_z \quad (3)$$

- Poisson equation

$$\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) + \frac{\partial}{\partial z}(3) \quad \Longrightarrow$$

$$\boxed{2\,U'(y)\,\frac{\partial v}{\partial x} = -\nabla^2 p + 2\,\nu_T'(y)\nabla^2 v + 2\,\nu_T''(y)\frac{\partial v}{\partial y} + \boldsymbol{\nabla}\cdot\boldsymbol{f}}$$

Linearized channel flow

▷ Orr-Sommerfeld equation: ∇^2 (2) + Poisson equation $\Longrightarrow$

$$\frac{\partial}{\partial t}(\nabla^2 v) + U(y)\nabla^2\left(\frac{\partial v}{\partial x}\right) - U''(y)\frac{\partial v}{\partial x} = \nu_T(y)\nabla^4 v + 2\nu_T'(y)\nabla^2\left(\frac{\partial v}{\partial y}\right) - \nu_T''(y)\nabla^2 v + 2\,\nu_T''(y)\frac{\partial^2 v}{\partial y^2} + \nabla^2 f_y - \frac{\partial}{\partial y}\boldsymbol{\nabla}\cdot\boldsymbol{f}$$

▷ Squire equation:

Reminder: $\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$,

$$\frac{\partial}{\partial z}(1) - \frac{\partial}{\partial x}(3) \quad \Longrightarrow$$

$$\left(\frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x}\right)\omega_y - \nu_T(y)\nabla^2\omega_y - \nu_T'(y)\frac{\partial}{\partial y}\omega_y - \frac{\partial f_x}{\partial z} + \frac{\partial f_z}{\partial x} = -U'(y)\frac{\partial v}{\partial z}$$

From 4 unknows (u, v, w, p) to 2 equations (Orr-Sommerfeld/Squire) with two unknowns $(v,\, \omega_y)$

Linearized channel flow

- Normal mode decomposition: $\boldsymbol{u}(\boldsymbol{x},t) = \hat{\boldsymbol{u}}(y,t)e^{i(\alpha x+\beta z)}$

  where α and β denote the streamwise and spanwise wavenumbers.
- Setting

$$k^2 = \alpha^2 + \beta^2 \quad \text{and} \quad \underline{\Delta} = \frac{\partial^2}{\partial y^2} - (\alpha^2+\beta^2) = D^2 - k^2 \quad \text{we show that}$$

$$\nabla^2 v = \underline{\Delta}\hat{v}(y,t)\,e^{\,i(\alpha x+\beta z)} \quad \text{and} \quad \frac{\partial}{\partial t}(\nabla^2 v) = \underline{\Delta}\dot{\hat{v}}(y,t)\,e^{\,i(\alpha x+\beta z)}$$

- After (some) manipulations, we obtain for Orr-Sommerfeld:

$$\boxed{\left(D^2-k^2\right)\frac{\partial}{\partial t}\hat{v} = \mathcal{L}_{OS}\hat{v} - \left(i\alpha D, k^2, i\beta D\right)\begin{pmatrix}\hat{f}_x\\ \hat{f}_y\\ \hat{f}_z\end{pmatrix}}$$

with

$$\mathcal{L}_{OS} = -i\alpha\left[U(D^2-k^2)-U''\right] + \nu_T(D^2-k^2)^2 + 2\,\nu_T'(D^2-k^2)D + \nu_T''(D^2+k^2)$$

▷ Similarly, we obtain for Squire:

$$\frac{\partial \hat{\omega}_y}{\partial t} = -i\beta U' \hat{v} + \mathcal{L}_{SQ}\, \hat{\omega}_y + (i\beta, 0, -i\alpha) \begin{pmatrix} \hat{f}_x \\ \hat{f}_y \\ \hat{f}_z \end{pmatrix}$$

with

$$\mathcal{L}_{SQ} = -i\alpha U + \nu_T (D^2 - k^2) + \nu_T' D$$

▷ State-space system

$$\begin{pmatrix} D^2 - k^2 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \hat{v} \\ \hat{\omega}_y \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{OS} & 0 \\ -i\beta U' & \mathcal{L}_{SQ} \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{\omega}_y \end{pmatrix} + \begin{pmatrix} -i\alpha D & -k^2 & -i\beta D \\ i\beta & 0 & -i\alpha \end{pmatrix} \begin{pmatrix} \hat{f}_x \\ \hat{f}_y \\ \hat{f}_z \end{pmatrix}$$

i.e.

$$E\frac{\partial}{\partial t}\begin{pmatrix}\hat{v}\\ \hat{\omega}_y\end{pmatrix} = A \quad \begin{pmatrix}\hat{v}\\ \hat{\omega}_y\end{pmatrix} + B_2\begin{pmatrix}\hat{f}_x\\ \hat{f}_y\\ \hat{f}_z\end{pmatrix}$$

▷ Observer.

Suppose we measure $\hat{\boldsymbol{y}} = (\hat{u}, \hat{v}, \hat{w})^T$. We then can show:

$$\begin{pmatrix}\hat{u}\\ \hat{v}\\ \hat{w}\end{pmatrix} = \frac{1}{k^2}\begin{pmatrix} i\alpha\frac{\partial}{\partial y} & -i\beta\\ k^2 & 0\\ i\beta\frac{\partial}{\partial y} & i\alpha\end{pmatrix}\begin{pmatrix}\hat{v}\\ \hat{\omega}_y\end{pmatrix} \quad \text{i.e.} \quad \hat{\boldsymbol{y}} = C_2\begin{pmatrix}\hat{v}\\ \hat{\omega}_y\end{pmatrix}$$

▷ Let $\hat{\boldsymbol{x}} = (\hat{v}, \hat{\omega}_y)^T$ and $\hat{\boldsymbol{u}} = \left(\hat{f}_x, \hat{f}_y, \hat{f}_z\right)^T$, we obtain:

$$E\frac{\partial}{\partial t}\hat{\boldsymbol{x}} = A\hat{\boldsymbol{x}} + B_2\hat{\boldsymbol{u}}$$

$$\hat{\boldsymbol{y}} = C_2\hat{\boldsymbol{x}} \qquad \text{State-space system}$$

Step #12: Example: Feedback control of linear systems with quadratic cost functions

LQ systems: Linear Quadratic systems

$$\frac{dq}{dt} = Aq + Bg \quad \text{with} \quad q(0) = q_0 \qquad \text{State equation}$$

Determine q and g such that

$$\mathcal{J}(q,g) = \frac{1}{2}\int_0^T \left(\underbrace{q \cdot Qq}_{\text{Performance}} + \gamma^2 \underbrace{g \cdot g}_{\text{Cost control}} \right) \mathrm{d}t$$

is minimized where Q is a symmetric positive definite matrix.

Ex: We consider the closed-loop solution of $\frac{dq}{dt} = Aq + Bg$ with $g = -Kq$ where K is the Kalman gain. We have directly:

$$\frac{dq}{dt} = (A - BK)\, q \qquad \text{closed-loop system}$$

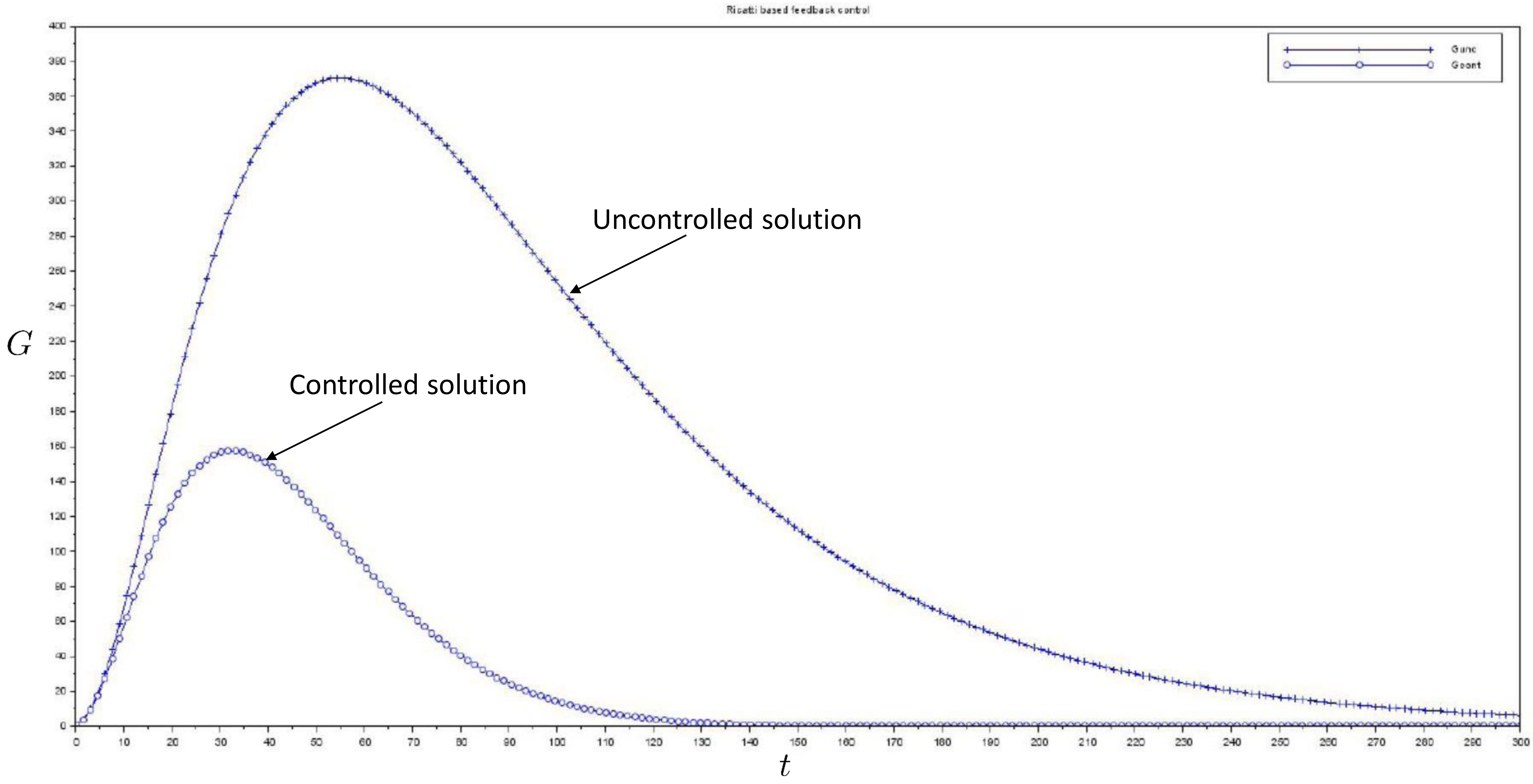
Scilab program: $A = \begin{bmatrix} -\frac{1}{\mathrm{Re}} & 0 \\ 1 & -\frac{3}{\mathrm{Re}} \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathrm{Re} = 100$.

Riccati based control.

LQR_Control.sce

```
// define system
Rey=100
A=[-1/Rey, 0; 1, -3/Rey]
B=[1, 0; 0, 1]
Q=[1, 0; 0, 1]
gam=1000
// solve Riccati equation
R=B*B'/gam^2;
X=riccati(A, R, Q, 'c', 'schur')
// compute feedback matrix
K=B'*X/gam^2
// linear modal stability of uncontrolled system
disp(spec(A))
// optimal transient energy growth of uncontrolled
// system is the L2 norm of exp(At)
Nt=200; t=linspace(0,3*Rey,Nt);
for j=1:Nt;
Gunc(j)=norm(expm(A*t(j)))^2;
end
Guncmax=max(Gunc)
// linear modal stability of controlled system
disp(spec(A-B*K))
// optimal transient energy growth of controlled
// system is the L2 norm of exp((A - BK)t)
for j=1:Nt
Gcont(j)=norm(expm((A - B*K)*t(j)))^2;
end
Gcontmax=max(Gcont)
// Plot
xtitle ( "Ricatti based feedback control" , "t" , "G" );
plot(t',Gunc,"+-");
plot(t',Gcont,"o-");
legend ( "Gunc" , "Gcont" );
//plot2d(t,[(Gunc),(Gcont)])
// end of program
```

Ricatti based control. Time evolution of $G(t)$.



Data Assimilation

Laurent Cordier

Master, 2021, Poitiers

Context

Meteorology

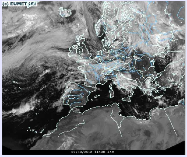
Model

$$\frac{\partial I}{\partial t} + \nabla I . \mathbf{v} = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} = \mathcal{N}(\mathbf{v})$$

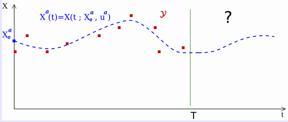
Titaud (2009)

Observations

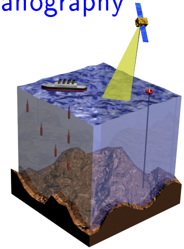


Météo France, 10/09 14h

Assimilation

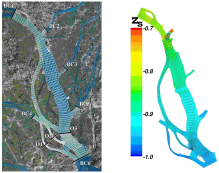


Oceanography



Daget (2008)

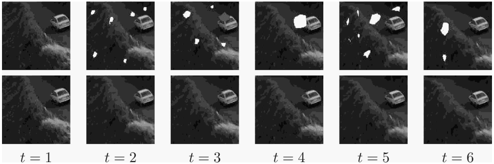
Floods forecast



Honnorat (2007)

Context

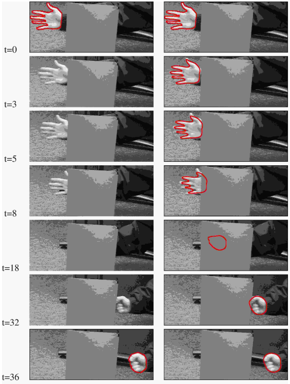
Movie restoration



Papadakis (2007)

Model: Level-Set/Lightness transport equation.
Control parameter: noise in the model.

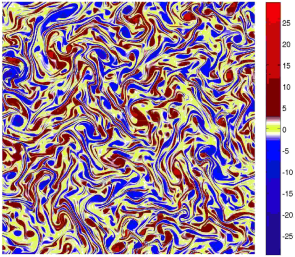
Contour tracking

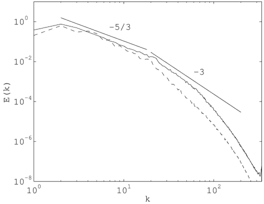


Papadakis (2007)

Context

Turbulence closure

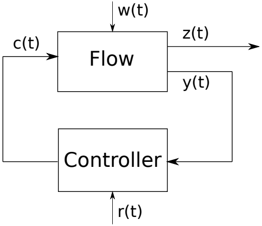


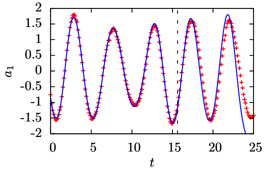


Farazmand & Protas (JFM, 2011)

Kato et al. (JCP, 2015)

Flow control





Context

Principle: To combine at best different sources of information to estimate (*optimally*) the state of a system:

- Imperfect observations (*incomplete, noised*)
- Imperfect model (*simplified*)
- *A priori* knowledge of the state of the system
  - background
  - statistics

Approaches:

- Variational method: Minimisation of a cost functional $\mathcal{J}$
- Stochastic method (*ex: Kalman Filter*)

Data-driven or *Physics-based*????

BOTH

Context

For what purpose?

- To identify initial and boundary conditions
- To identify unknown model parameters
- To improve numerical modelling (Turbulence model for instance)
- To interpolate optimally sparse observations
- To reconstruct indirectly non observed variables (virtual sensor)

Chapter 1

Data Assimilation

1.1 Error statistics

Mean:

$\mathbb{E}(x) = \langle x \rangle$ scalar ; $\mathbb{E}(\mathbf{x}) = (\mathbb{E}(x_1), \mathbb{E}(x_2), \cdots, \mathbb{E}(x_n))$ vector-valued

Variance, covariance (x, y scalars):

$$\mathrm{Var}(x) = \mathbb{E}\left((x - \mathbb{E}(x))^2\right) \quad ; \quad \mathrm{Cov}(x, y) = \mathbb{E}\left((x - \mathbb{E}(x))(y - \mathbb{E}(y))\right)$$

We say that errors ϵ are:

- unbiased if $\mathbb{E}(\epsilon) = 0$;
- uncorrelated if $\mathbb{E}\left(\epsilon_1 \epsilon_2^T\right) = 0$ (errors are independent);
- non trivial if $\mathrm{Cov}(\epsilon)$ is positive definite[1].

Covariance matrix ($\mathbf{x}$ vector-valued):

$$\mathrm{Cov}(\mathbf{x}) = \mathbb{E}\left((\mathbf{x} - \mathbb{E}(\mathbf{x}))(\mathbf{x} - \mathbb{E}(\mathbf{x}))^T\right)$$
$$(\mathrm{Cov}(\mathbf{x}))_{i,j} = \mathrm{Cov}(x_i, x_j) = \mathbb{E}\left((x_i - \mathbb{E}(x_i))(x_j - \mathbb{E}(x_j))\right)$$

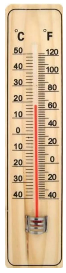
e.g. for $\mathbf{x} = (x_1, x_2, x_3)$:

$$\mathrm{Cov}(\mathbf{x}) = \begin{pmatrix} \mathrm{Var}(x_1) & \mathrm{Cov}(x_1, x_2) & \mathrm{Cov}(x_1, x_3) \\ \mathrm{Cov}(x_1, x_2) & \mathrm{Var}(x_2) & \mathrm{Cov}(x_2, x_3) \\ \mathrm{Cov}(x_1, x_3) & \mathrm{Cov}(x_2, x_3) & \mathrm{Var}(x_3) \end{pmatrix}$$

[1] M positive-definite $\iff$ $\mathbf{x}^* M \mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$

1.2 A very simple scalar example

Suppose we have a **background** estimate of the temperature in a room T_b and a measurement (**observation**) of the temperature T_o. We assume that these estimates are unbiased and uncorrelated.



(a) T_o

(b) T_b

What is the best estimate of the true temperature T_t?

Example: Two observations $T_o = 1$ and $T_b = 2$ of some unknown quantity T.

$$\text{Min}_T \left[(T-1)^2 + (T-2)^2 \right] \Longrightarrow \hat{T} = \frac{3}{2}$$

Problems:

1. Sensitivity to any change of unit.
   1 obs. $T_o = 1$ of T and 1 obs. $T_b = 4$ of $2 \times T$

$$\text{Min}_T \left[(T-1)^2 + (2T-4)^2 \right] \Longrightarrow \hat{T} = \frac{9}{5}$$

2. No sensitivity to the precision of the measurement: same estimate even if T_o is more accurate that T_b.

We consider the best estimate (**analysis**) to be a linear combination of the background and measurement.

$$T_a = \alpha_b T_b + \alpha_o T_o$$

Then the question is how should we choose α_b and α_o?
We need to impose two conditions.

1. We want the analysis to be unbiased.

   Let

$$\begin{aligned} T_a &= T_t + \epsilon_a \\ T_b &= T_t + \epsilon_b \\ T_o &= T_t + \epsilon_o \end{aligned}$$

   Then

$$\begin{aligned} \langle \epsilon_a \rangle &= \langle T_a - T_t \rangle \\ &= \langle \alpha_b T_b + \alpha_o T_o - T_t \rangle \\ &= \langle \alpha_b (T_b - T_t) + \alpha_o (T_o - T_t) + (\alpha_b + \alpha_o - 1) T_t \rangle \\ &= \alpha_b \langle \epsilon_b \rangle + \alpha_o \langle \epsilon_o \rangle + (\alpha_b + \alpha_o - 1) \langle T_t \rangle \end{aligned}$$

   Hence to ensure that $\langle \epsilon_a \rangle = 0$ for all values of T_t, we require that

$$\alpha_b + \alpha_o = 1$$

   so

$$\boxed{T_a = \alpha_b T_b + (1 - \alpha_b) T_o}$$

2. We want the uncertainty in our analysis to be as small as possible *i.e.* we want to minimize the variance.

   Let

$$\begin{aligned} \langle \epsilon_a^2 \rangle &= \sigma_a^2 \\ \langle \epsilon_b^2 \rangle &= \sigma_b^2 \\ \langle \epsilon_o^2 \rangle &= \sigma_o^2 \end{aligned}$$

   Then

$$\begin{aligned} \sigma_a^2 &= \langle (T_a - T_t)^2 \rangle \\ &= \langle (\alpha_b T_b + (1 - \alpha_b) T_o - T_t)^2 \rangle \\ &= \langle (\alpha_b (T_b - T_t) + (1 - \alpha_b)(T_o - T_t))^2 \rangle \\ &= \langle (\alpha_b \epsilon_b + (1 - \alpha_b) \epsilon_o)^2 \rangle \\ &= \alpha_b^2 \sigma_b^2 + (1 - \alpha_b)^2 \sigma_o^2 \quad \text{using} \quad \langle \epsilon_b \epsilon_o \rangle = 0 \end{aligned}$$

Then setting $\dfrac{d\sigma_a^2}{d\alpha_b} = 0$, we find

$$\alpha_b = \frac{\sigma_o^2}{\sigma_o^2 + \sigma_b^2}$$

Hence we have

$$T_a = \frac{\sigma_o^2}{\sigma_o^2 + \sigma_b^2} T_b + \frac{\sigma_b^2}{\sigma_o^2 + \sigma_b^2} T_o = \frac{\dfrac{1}{\sigma_b^2} T_b + \dfrac{1}{\sigma_o^2} T_o}{\dfrac{1}{\sigma_b^2} + \dfrac{1}{\sigma_o^2}}$$

This is known as the Best Linear Unbiased Estimate (BLUE).

We also find that

1. $$\sigma_a^2 = \frac{\sigma_b^2 \sigma_o^2}{\sigma_o^2 + \sigma_b^2} < \min\{\sigma_b^2, \sigma_o^2\}$$

2. Let p be the precision:

$$p(T_a) \triangleq \frac{1}{\mathrm{Var}(T_a)} = \frac{1}{\sigma_b^2} + \frac{1}{\sigma_o^2} = p(T_b) + p(T_o) \quad \text{Precisions are added}$$

Variational point of view. Looking for the BLUE is equivalent to solving:

$$\text{Minimize}_T \quad J(T) = J_b(T) + J_o(T) = \frac{1}{2}\left[\frac{(T - T_b)^2}{\sigma_b^2} + \frac{(T - T_o)^2}{\sigma_o^2}\right]$$

Remarks:

- This gives a rationale for the choice of the norm in J.
- This solves the problem of sensitivity to the units and non-sensitivity to the precisions.
- $J''(x) = \dfrac{1}{\sigma_b^2} + \dfrac{1}{\sigma_o^2} = \dfrac{1}{\mathrm{Var}(T_a)}$
  Convexity of $J \rightarrow$ precision of the estimate.

Alternative formulation: background + observation. The analysed temperature reads:

$$T_a = T_b + \underbrace{\frac{\sigma_b^2}{\sigma_b^2 + \sigma_o^2}}_{Gain} \underbrace{(T_o - T_b)}_{Innovation}$$

Remark: When everything is linear, the BLUE is equivalent to solve a variational problem.

How can we generalize this to a vector state and a vector of observations?

1.3 BLUE for vectorial quantities

We consider:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \epsilon^o \qquad \mathbf{x}^b = \mathbf{x} + \epsilon^b$$

where $\mathbf{H}$ is linear. We assume the Gaussian approximation, *i.e.*

$$\epsilon^o \sim \mathcal{N}(\mathbf{0}, \mathbf{R}) \quad \text{and} \quad \epsilon^b \sim \mathcal{N}(\mathbf{0}, \mathbf{B})$$

Analysed solution:

$$\begin{aligned} \mathbf{x}^a &= \mathbf{x}^b + \mathbf{K}\left(\mathbf{y} - \mathbf{H}\mathbf{x}^b\right) \\ \mathbf{K} &= \mathbf{B}\mathbf{H}^T\left(\mathbf{R} + \mathbf{H}\mathbf{B}\mathbf{H}^T\right)^{-1} \\ \mathbf{P}^a &= (\mathbf{I} - \mathbf{K}\mathbf{H})\,\mathbf{B} \end{aligned}$$

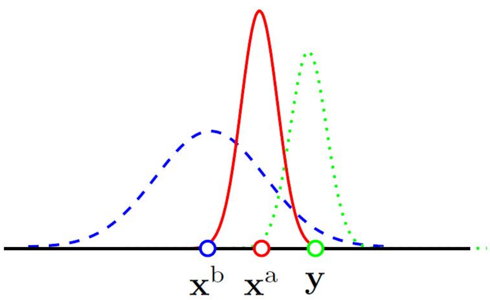


Figure 1.1: PdF.

Error statistics. Assumptions and definitions.

- $\mathbf{x}^t$ is defined as the true unknown state.

- Observation error statistics:

$$\epsilon^o = \mathbf{y} - \mathbf{H}\mathbf{x}^t \quad \text{with} \quad \mathbb{E}\left[\epsilon^o\right] = \mathbf{0} \qquad \mathbb{E}\left[\epsilon^o \epsilon^{oT}\right] = \mathbf{R}$$

which is in particular satisfied if $\epsilon^o \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$.

- Background error statistics:

$$\epsilon^b = \mathbf{x}^b - \mathbf{x}^t \quad \text{with} \quad \mathbb{E}\left[\epsilon^b\right] = \mathbf{0} \qquad \mathbb{E}\left[\epsilon^b \epsilon^{bT}\right] = \mathbf{B} \qquad \mathbb{E}\left[\epsilon^b \epsilon^{oT}\right] = \mathbf{0}$$

- Analysis error statistics:

$$\epsilon^a = \mathbf{x}^a - \mathbf{x}^t \quad \text{with} \quad \mathbb{E}\left[\epsilon^a\right] = \mathbf{0} \qquad \mathbb{E}\left[\epsilon^a \epsilon^{aT}\right] = \mathbf{P}^a$$

Linear unbiased Ansatz for the estimate

- General Ansatz, linear in the first guess and the observation

$$\mathbf{x}^a = \mathbf{L}\mathbf{x}^b + \mathbf{K}\mathbf{y}$$

- Writing it in terms of errors

$$\begin{aligned}\epsilon^a = \mathbf{x}^a - \mathbf{x}^t &= \mathbf{L}\left(\mathbf{x}^b - \mathbf{x}^t + \mathbf{x}^t\right) + \mathbf{K}\left(\mathbf{H}\mathbf{x}^t + \epsilon^o\right) - \mathbf{x}^t \\ &= \mathbf{L}\epsilon^b + \mathbf{K}\epsilon^o + \left(\mathbf{L} + \mathbf{K}\mathbf{H} - \mathbf{I}\right)\mathbf{x}^t\end{aligned}$$

Then $\mathbb{E}\left[\epsilon^o\right] = \mathbf{0}$ and $\mathbb{E}\left[\epsilon^b\right] = \mathbf{0}$ imply $\mathbb{E}\left[\epsilon^a\right] = \left(\mathbf{L} + \mathbf{K}\mathbf{H} - \mathbf{I}\right)\mathbb{E}\left[\mathbf{x}^t\right] = \mathbf{0}$. Hence, we wish to impose

$$\boxed{\mathbf{L} = \mathbf{I} - \mathbf{K}\mathbf{H}}$$

As a result, we obtain a linear and unbiased Ansatz:

$$\begin{aligned}\mathbf{x}^a &= \left(\mathbf{I} - \mathbf{K}\mathbf{H}\right)\mathbf{x}^b + \mathbf{K}\mathbf{y} \\ &= \mathbf{x}^b + \mathbf{K}\underbrace{\left(\mathbf{y} - \mathbf{H}\mathbf{x}^b\right)}_{\text{Innovation}}\end{aligned}$$

Best linear unbiased estimator

- Posterior error:

$$\begin{aligned}\epsilon^a &= \mathbf{L}\epsilon^b + \mathbf{K}\epsilon^o \\ &= \epsilon^b + \mathbf{K}\left(\epsilon^o - \mathbf{H}\epsilon^b\right)\end{aligned}$$

so that

$$\begin{aligned}\mathbf{P}^a &= \mathbb{E}\left[\epsilon^a \epsilon^{aT}\right] = \mathbb{E}\left[\left(\epsilon^b + \mathbf{K}\left(\epsilon^o - \mathbf{H}\epsilon^b\right)\right)\left(\epsilon^b + \mathbf{K}\left(\epsilon^o - \mathbf{H}\epsilon^b\right)\right)^T\right] \\ &= \mathbb{E}\left[\left(\mathbf{L}\epsilon^b + \mathbf{K}\epsilon^o\right)\left(\mathbf{L}\epsilon^b + \mathbf{K}\epsilon^o\right)^T\right] \\ &= \mathbb{E}\left[\mathbf{L}\epsilon^b\left(\epsilon^b\right)^T\mathbf{L}^T\right] + \mathbb{E}\left[\mathbf{K}\epsilon^o\left(\epsilon^o\right)^T\mathbf{K}^T\right] \\ &= \mathbf{LBL}^T + \mathbf{KRK}^T\end{aligned}$$

In summary:

$$\boxed{\mathbf{P}^a = \left(\mathbf{I} - \mathbf{KH}\right)\mathbf{B}\left(\mathbf{I} - \mathbf{KH}\right)^T + \mathbf{KRK}^T}$$

- We look for a global measure of the error, for instance $\mathrm{Tr}\left(\mathbf{P}^a\right)$. Let us find the optimal $\mathbf{K}$ that minimizes this metric.

Best linear unbiased estimator

- Variation of the metric with respect to a variation of $\mathbf{K}$, *i.e.* $\delta\mathbf{K}$

$$\begin{aligned}\delta\left(\mathrm{Tr}\left(\mathbf{P}^a\right)\right) &= \mathrm{Tr}\left(\left(-\delta\mathbf{KH}\right)\mathbf{BL}^T + \mathbf{LB}\left(-\delta\mathbf{KH}\right)^T + \delta\mathbf{KRK}^T + \mathbf{KR}\delta\mathbf{K}^T\right) \\ &= \mathrm{Tr}\left(\left(-\mathbf{LB}^T\mathbf{H}^T - \mathbf{LBH}^T + \mathbf{KR}^T + \mathbf{KR}\right)\left(\delta\mathbf{K}\right)^T\right) \\ &= 2\mathrm{Tr}\left(\left(-\mathbf{LBH}^T + \mathbf{KR}\right)\left(\delta\mathbf{K}\right)^T\right)\end{aligned}$$

- At optimality, one infers that

$$\begin{aligned}-\mathbf{LBH}^T + \mathbf{K}^\star\mathbf{R} &= \mathbf{0} \\ -\left(\mathbf{I} - \mathbf{K}^\star\mathbf{H}\right)\mathbf{BH}^T + \mathbf{K}^\star\mathbf{R} &= \mathbf{0}\end{aligned}$$

, from which we obtain:

$$\mathbf{K}^\star = \mathbf{BH}^T\left(\mathbf{R} + \mathbf{HBH}^T\right]^{-1}$$

from which we get the BLUE solution:

$$\begin{aligned}\mathbf{x}^a &= \mathbf{x}^b + \mathbf{K}\left(\mathbf{y} - \mathbf{Hx}^b\right) \\ \mathbf{K} &= \mathbf{BH}^T\left(\mathbf{R} + \mathbf{HBH}^T\right)^{-1} \\ \mathbf{P}^a &= \left(\mathbf{I} - \mathbf{KH}\right)\mathbf{B}\end{aligned}$$

Inserting the expression of $\mathbf{K}^\star$ onto the one of $\mathbf{P}^a$, we get:

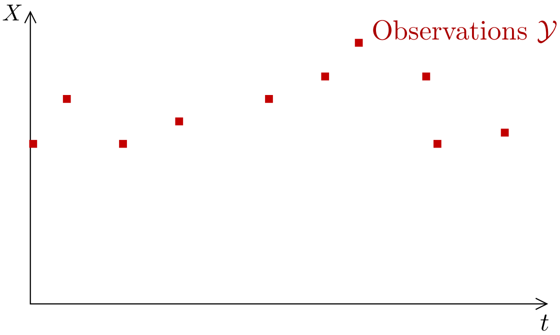
$$\mathbf{P}^a = (\mathbf{I} - \mathbf{K}^\star\mathbf{H})\,\mathbf{B}$$

4D-Var identification

Principle

Papadakis (2007)

We have some observations $\mathcal{Y}$.



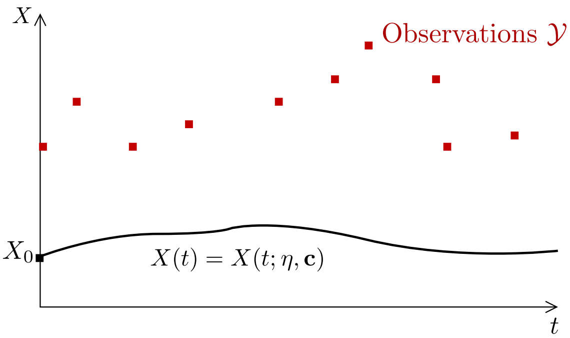
4D-Var identification

Principle

Papadakis (2007)

We know a model $\mathbb{M}$. $\quad$ c model parameters

$$\frac{\partial X(t)}{\partial t} + \mathbb{M}(X(t), c) = 0 \quad ; \quad X(0) = X_0$$



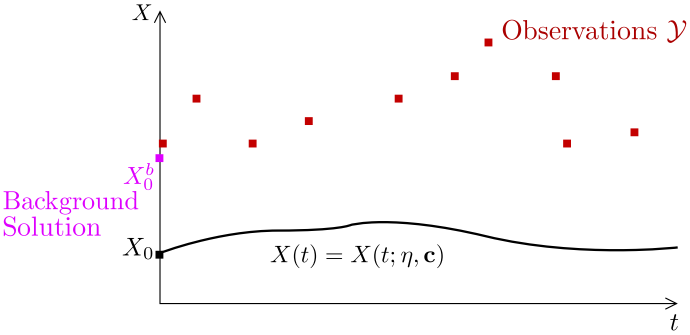
4D-Var identification

Principle

Papadakis (2007)

We know a background solution $(X_0^b, \mathbf{c}^b)$. $(\eta, \mathbf{c})$ control parameters.

$$\frac{\partial X(t)}{\partial t} + \mathbb{M}(X(t), \mathbf{c}) = 0 \quad ; \quad X(0) = X_0^b + \eta$$



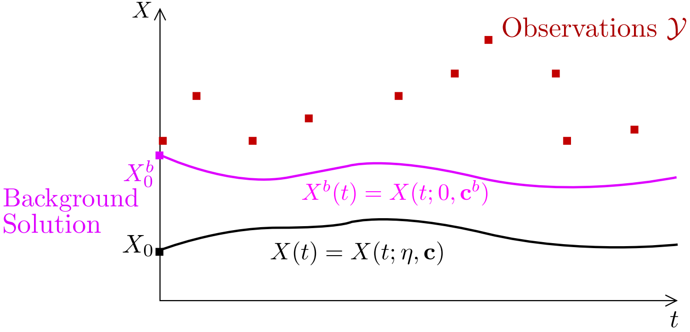
4D-Var identification

Principle

Papadakis (2007)

We know a background solution $(X_0^b, \mathbf{c}^b)$. $(\eta, \mathbf{c})$ control parameters.

$$\frac{\partial X(t)}{\partial t} + \mathbb{M}(X(t), \mathbf{c}) = 0 \quad ; \quad X(0) = X_0^b + \eta$$



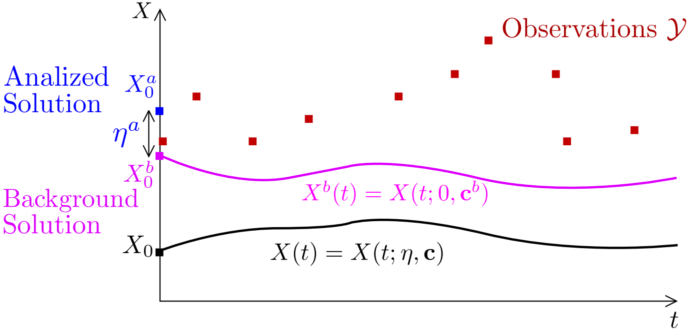
4D-Var identification

Principle

Papadakis (2007)

4D-Var: search for $(\eta^a, \mathbf{c}^a) = \text{argmin}(\mathcal{J}(\eta, \mathbf{c}))$

$$\mathcal{J}(\eta, \mathbf{c}) = \frac{1}{2}\int_0^T \|\mathcal{Y} - \mathbb{H}(X(t;\eta,\mathbf{c}))\|^2_{\mathcal{R}^{-1}}\,\mathrm{d}t + \frac{1}{2}\|\eta\|^2_{\mathcal{B}^{-1}} + \frac{1}{2}\|\mathbf{c} - \mathbf{c}^b\|^2_{\mathcal{C}^{-1}}$$



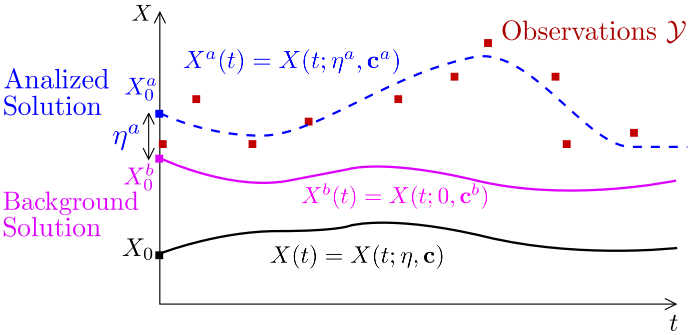
4D-Var identification

Principle

Papadakis (2007)

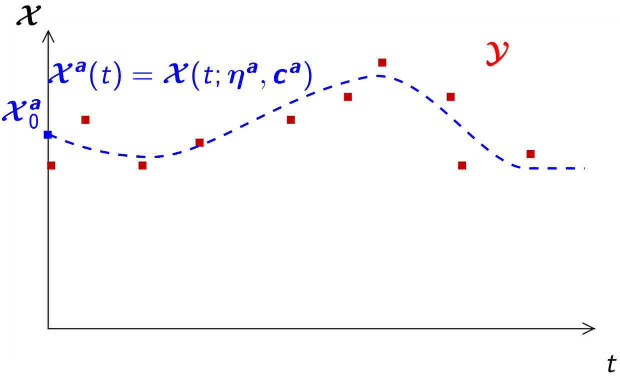
4D-Var: search for $(\eta^a, \boldsymbol{c}^a) = \text{argmin}(\mathcal{J}(\eta, \boldsymbol{c}))$

$$\mathcal{J}(\eta, \boldsymbol{c}) = \frac{1}{2}\int_0^T \|\mathcal{Y} - \mathbb{H}(X(t;\eta,\boldsymbol{c}))\|^2_{\mathcal{R}^{-1}}\,\mathrm{d}t + \frac{1}{2}\|\eta\|^2_{\mathcal{B}^{-1}} + \frac{1}{2}\|\boldsymbol{c} - \boldsymbol{c}^b\|^2_{\mathcal{C}^{-1}}$$



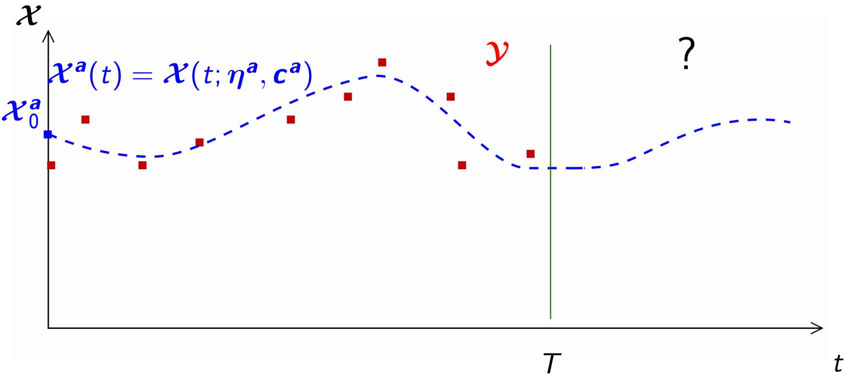
Variational approach

Objectives of Data Assimilation: Estimate



Variational approach

Objectives of Data Assimilation: Forecast



Variational approach 4D-Var formalism *Papadakis (2007)*

Model:

$$\begin{cases} \dfrac{\partial \boldsymbol{\mathcal{X}}(t)}{\partial t} + \mathbb{M}(\boldsymbol{\mathcal{X}}(t), \boldsymbol{c}) = 0 \\ \boldsymbol{\mathcal{X}}(0) = \boldsymbol{\mathcal{X}}_0 + \boldsymbol{\eta} \end{cases}$$

Cost functional:

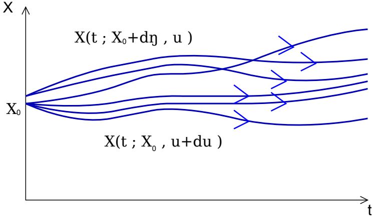
$$\mathcal{J}(\boldsymbol{\eta}, \boldsymbol{c}) = \frac{1}{2}\int_0^T \|\boldsymbol{\mathcal{Y}} - \mathbb{H}(\boldsymbol{\mathcal{X}}(t; \boldsymbol{\eta}, \boldsymbol{c}))\|^2_{\mathcal{R}^{-1}}\,\mathrm{d}t + \frac{1}{2}\|\boldsymbol{\eta}\|^2_{\mathcal{B}^{-1}} + \frac{1}{2}\|\boldsymbol{c} - \boldsymbol{c}^{\boldsymbol{b}}\|^2_{\mathcal{C}^{-1}}$$

- $\mathbb{H}$ observation operator
- $\mathcal{R}$, $\mathcal{B}$ and $\mathcal{C}$ correlation matrix which represent how we trust in the observations and the background solutions.

Variational approach 4D-Var formalism Gradient computation

Descent algorithm, we need $\nabla \mathcal{J}$:

Finite differences

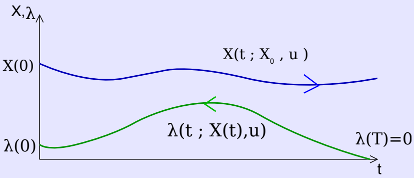


$$\left(\frac{\partial \mathcal{J}}{\partial \boldsymbol{\eta}}, \delta \boldsymbol{\eta}\right) = \frac{\mathcal{J}(\boldsymbol{\eta}+\epsilon\delta\boldsymbol{\eta}, \boldsymbol{c}) - \mathcal{J}(\boldsymbol{\eta}, \boldsymbol{c})}{\epsilon}$$

$$\left(\frac{\partial \mathcal{J}}{\partial \boldsymbol{c}}, \delta \boldsymbol{c}\right) = \frac{\mathcal{J}(\boldsymbol{\eta}, \boldsymbol{c}+\epsilon\delta\boldsymbol{c}) - \mathcal{J}(\boldsymbol{\eta}, \boldsymbol{c})}{\epsilon}$$

$N_c + 1$ temporal integrations

Adjoint method



$\nabla \mathcal{J}$ is a function of $\boldsymbol{\lambda}(t)$

2 temporal integrations

Variational approach 4D-Var formalism Optimality system

Find $\nabla \mathcal{J}$ for minimisation: $\boldsymbol{\lambda}(t)$ Lagrange multiplier

- Adjoint equation

$$\begin{cases} -\dfrac{\partial \boldsymbol{\lambda}}{\partial t}(t) + \left(\dfrac{\partial \mathbb{M}}{\partial \boldsymbol{\mathcal{X}}}\right)^{+} \boldsymbol{\lambda}(t) = \left(\dfrac{\partial \mathbb{H}}{\partial \boldsymbol{\mathcal{X}}}\right)^{+} \mathcal{R}^{-1} \left(\mathbb{H}(\boldsymbol{\mathcal{X}}(t)) - \boldsymbol{\mathcal{Y}}\right) \\ \boldsymbol{\lambda}(T) = 0 \end{cases}$$

- Optimality condition

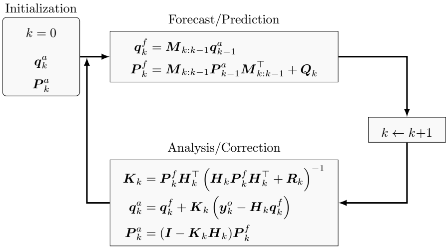
$$\begin{cases} \dfrac{\partial \mathcal{J}}{\partial \boldsymbol{\eta}} = \boldsymbol{\lambda}(0) + \mathcal{B}^{-1} \boldsymbol{\eta} \\ \dfrac{\partial \mathcal{J}}{\partial \boldsymbol{c}} = -\displaystyle\int_0^T \left(\dfrac{\partial \mathbb{M}}{\partial \boldsymbol{c}}\right)^{+} \boldsymbol{\lambda}(t)\, \mathrm{d}t + \mathcal{C}^{-1}(\boldsymbol{c} - \boldsymbol{c}^{\boldsymbol{b}}) \end{cases}$$

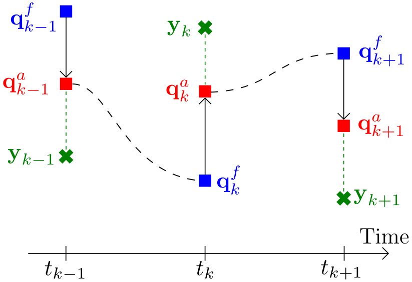
Kalman filter

Generality

Bayesian formalism: propagation of the mean and covariance of the state through time.

Linear model: $\boldsymbol{q}_k = \boldsymbol{M}_{k:k-1}\boldsymbol{q}_{k-1} + \boldsymbol{\eta}_k, \quad \boldsymbol{\eta}_k \sim \mathcal{N}(0, \boldsymbol{Q}_k)$.
Observation: $\boldsymbol{y}_k^o = \boldsymbol{H}_k\boldsymbol{q}_k + \boldsymbol{\epsilon}_k^o, \quad \boldsymbol{\epsilon}_k^o \sim \mathcal{N}(0, \boldsymbol{R}_k)$.





Difficult to use in Fluid Mechanics: nonlinearity, size of the states, etc.

Ensemble Kalman filter (EnKF)

Generality

EnKF uses the Monte Carlo method to empirically represent the statistics of the estimator.

Non linear model and observations.

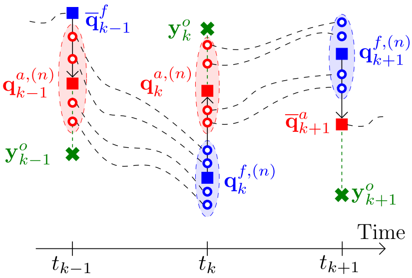
$$\begin{cases} \boldsymbol{q}_k^t = \mathcal{M}_{k:k-1}(\boldsymbol{q}_{k-1}) + \boldsymbol{\eta}_k, & \boldsymbol{\eta}_k \sim \mathcal{N}(0, \boldsymbol{Q}_k), \\ \boldsymbol{y}_k^o = \mathcal{H}_k(\boldsymbol{q}_k) + \boldsymbol{\epsilon}_k^o, & \boldsymbol{\epsilon}_k^o \sim \mathcal{N}(0, \boldsymbol{R}_k). \end{cases}$$

$\boldsymbol{P}_k^f$ obtained as $\boldsymbol{P}_k^{f,e}$

$$\boldsymbol{P}_k^{f,e} = \frac{1}{N_e - 1}\sum_{n=1}^{N_e}(\boldsymbol{q}_k^{f,(n)} - \overline{\boldsymbol{q}}_k^f)(\boldsymbol{q}_k^{f,(n)} - \overline{\boldsymbol{q}}_k^f)^\top,$$

where

$$\overline{\boldsymbol{q}}_k^f = \frac{1}{N_e}\sum_{n=1}^{N_e}\boldsymbol{q}_k^{f,(n)}.$$



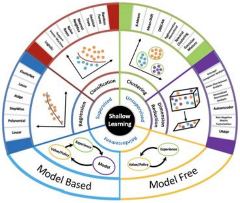
Extension to Dual EnKF: for estimating the states $\boldsymbol{q}_k$ and model parameters $\boldsymbol{\Theta}_k$ iteratively.

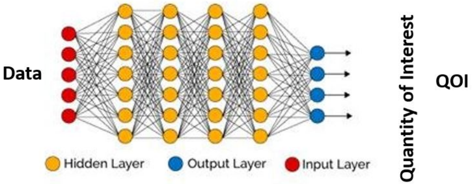
Machine Learning

Definitions and Applications

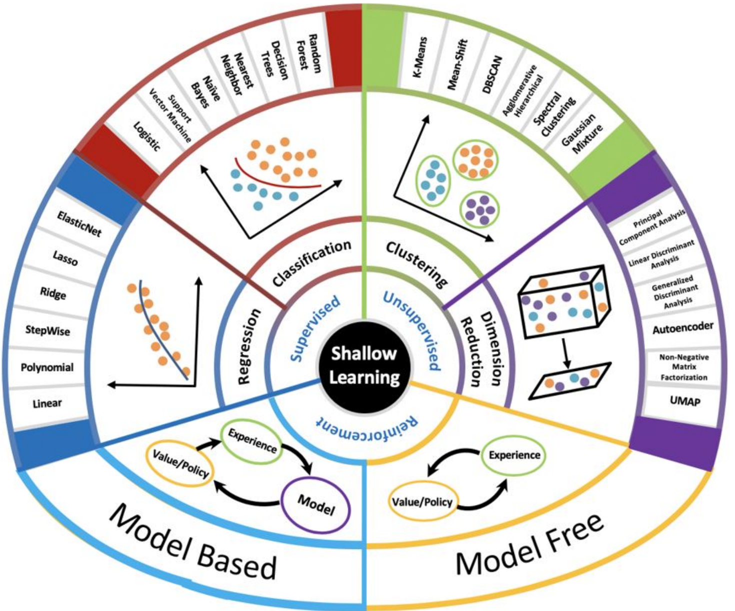
> "Field of study that gives computers the ability to learn without being explicitly programmed."
>
> — Arthur Samuel (1959)

- Machine Learning techniques, and in particular Deep Learning (DL), have recently demonstrated impressive skills in reproducing complex spatiotemporal dynamics.
- The emergence of DL is largely due to:
  - the development of efficient and user-friendly libraries (Python);
  - the increasing computation capabilities (GPUs, TPUs);
  - the access to (very) large datasets for training.





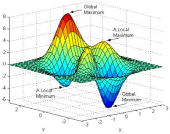
Machine Learning techniques



> "Machine learning (ML) algorithms build a model based on sample data, known as "training data", in order to make predictions without being explicitly programmed to perform the task."

- In most cases, the goal is to minimize a cost function which expresses the discrepancy between the model prediction and the data:

$$\boldsymbol{w}^* = \underset{\boldsymbol{w}\in\mathbb{R}^{N_p}}{\arg\min} \sum_{i=1}^{N_e} \|\boldsymbol{y}_i - \mathcal{M}(\boldsymbol{w}, \boldsymbol{x}_i)\|_2^2 \,.$$



  - The set $\{(\boldsymbol{x}_i, \boldsymbol{y}_i)\,, i = 1, \cdots, N_e\}$ is the trainig data.
  - The model $\mathcal{M}$ is parametrized by a set of parameters $\boldsymbol{w} \in \mathbb{R}^{N_p}$.
- This approach is called supervised learning.
- In this sense, ML is not far away from data assimilation (DA).

- The cost function to minimise in 4D-Var is:

$$\mathcal{J}(\boldsymbol{w}, \boldsymbol{x}_0, \cdots, \boldsymbol{x}_{N_t}) = \frac{1}{2} \sum_{k=0}^{N_t} \|\boldsymbol{y}_k - \mathcal{H}_k(\boldsymbol{x}_k)\|^2_{\boldsymbol{R}_k^{-1}} + \frac{1}{2} \sum_{k=0}^{N_t-1} \|\boldsymbol{x}_{k+1} - \mathcal{M}_k(\boldsymbol{w}, \boldsymbol{x}_k)\|^2_{\boldsymbol{Q}_k^{-1}}$$

  - $\boldsymbol{x}_k \in \mathbb{R}^{N_x}$ is the state at time t_k;
  - $\boldsymbol{y}_k \in \mathbb{R}^{N_y}$ is the observation vector at time t_k;
  - $\boldsymbol{w} \in \mathbb{R}^{N_p}$ is the set of parameters of the model $\mathcal{M}_k$ (e.g., the weights of an artificial neural network);
  - N_t is the length of the assimilation or training window.
- If $\mathcal{H}_k = \boldsymbol{I}$ (full observations) and $\boldsymbol{R} = 0$ (no observation noise), we recover the standard ML cost function.

- Suppose that $\boldsymbol{f}(t)$ is the trajectory of a physical system. Define:

$$\boldsymbol{x}_i = \boldsymbol{f}(i \times \Delta t)$$
$$\boldsymbol{y}_i = \boldsymbol{f}((i+1) \times \Delta t).$$

- Then the ML minimization problem consists in finding the best approximation of the map

$$\boldsymbol{f}(t) \mapsto \boldsymbol{f}(t + \Delta t)$$

i.e., the resolvent of $\boldsymbol{f}(t)$, among all models

$$\left\{ \mathcal{M} : \boldsymbol{x} \mapsto \mathcal{M}(\boldsymbol{w}, \boldsymbol{x}), \boldsymbol{w} \in \mathbb{R}^{N_p} \right\}.$$

- Instead of constructing the model from scratch, we could build a hybrid model using an already existent model:

$$\mathcal{M}_k^h : (\boldsymbol{w}, \boldsymbol{x}) \mapsto \mathcal{M}_k^o + \mathcal{M}_k^{ml}$$

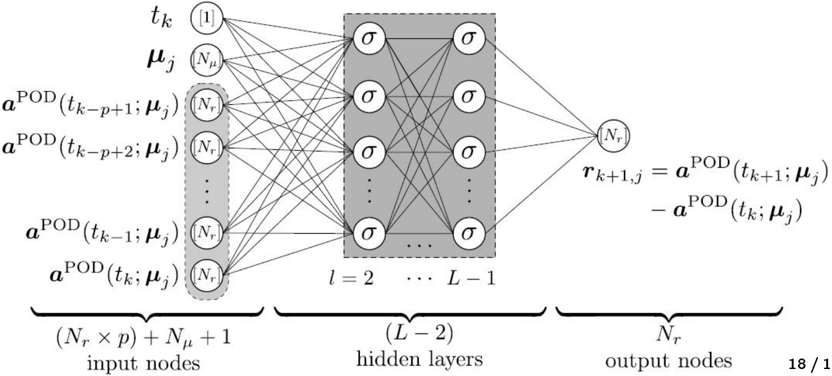
where $\mathcal{M}^o$ is the original model and $\mathcal{M}_k^{ml}$ is the trainable model.

Neural Network Reduced-Order Model

$$\boldsymbol{r}(t_{k+1};\boldsymbol{\mu}_j) = \boldsymbol{f}^{\mathrm{NN}}\left(\boldsymbol{a}^{\mathrm{POD}}(t_i;\boldsymbol{\mu}_j)_{i=k-p+1}^{k};\boldsymbol{W},\boldsymbol{b}\right) + \epsilon_{k+1}$$

$$\boldsymbol{r}(t_{k+1};\boldsymbol{\mu}_j) = \boldsymbol{a}^{\mathrm{POD}}(t_{k+1};\boldsymbol{\mu}_j) - \boldsymbol{a}^{\mathrm{POD}}(t_k;\boldsymbol{\mu}_j)$$

$$\boldsymbol{f}^{\mathrm{NN}}(\boldsymbol{W},\boldsymbol{b}) : \{\boldsymbol{a}^{\mathrm{POD}}(t_i;\boldsymbol{\mu}_j)\}_{i=k-p+1}^{k} \rightarrow \{\boldsymbol{r}(t_{k+1};\boldsymbol{\mu}_j)\}$$
$$\boldsymbol{a}^{\mathrm{POD}} \in \mathbb{R}^{N_r\times 1};\quad \boldsymbol{\mu}_j \in \mathbb{R}^{N_\mu\times 1}$$




QUESTIONS??
Fish '11