INSTITUT PPRIME
CNRS–UPR–3346 • UNIVERSITÉ DE POITIERS • ENSMA

DÉPARTEMENT D2 – FLUIDES
THERMIQUE ET COMBUSTION

An introduction to hydrodynamic stability

Lecture 7: beyond the critical point

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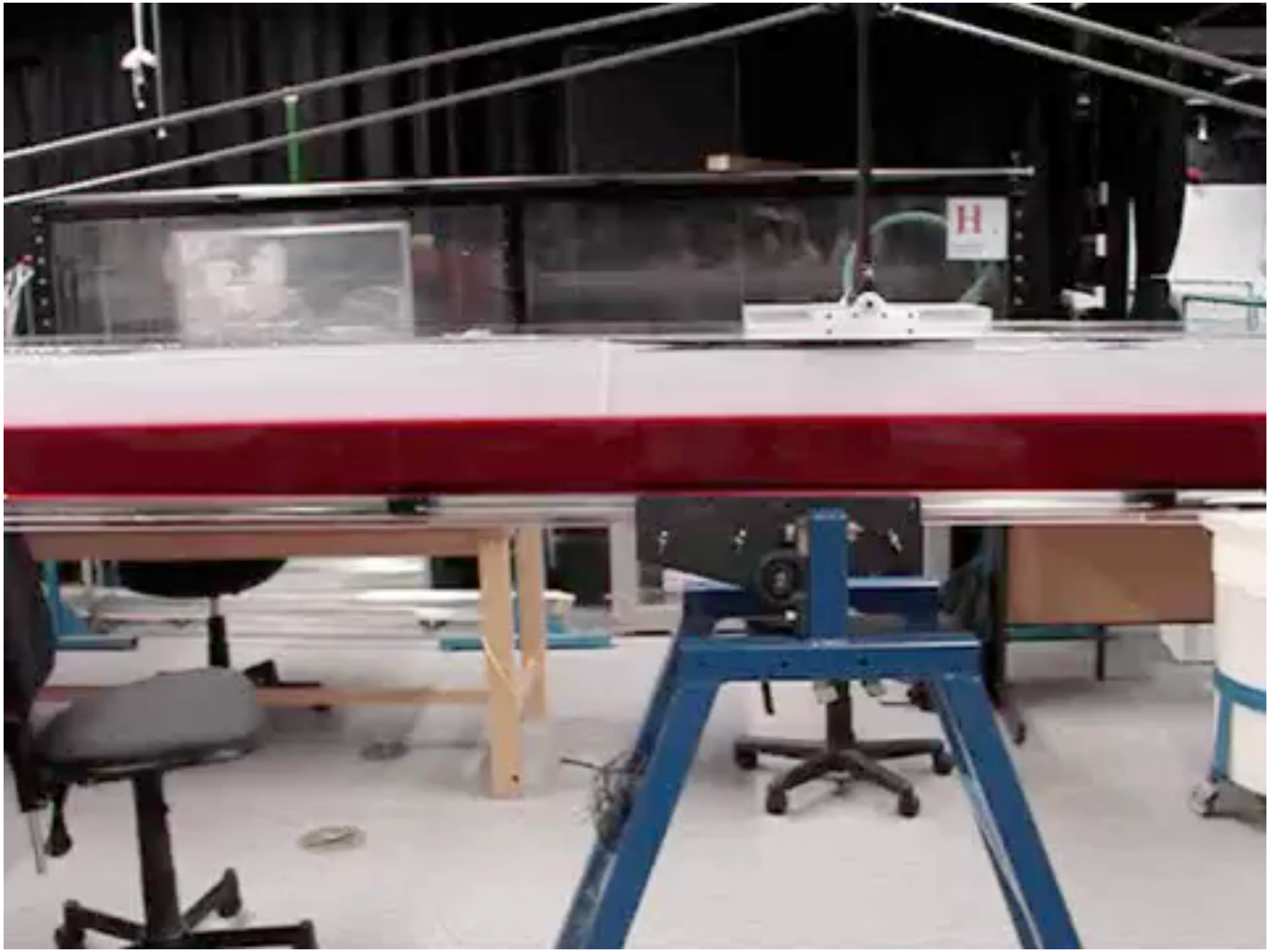
1. Beyond the critical point?
2. State-space representation of dynamical systems
3. Local bifurcation theory

* Part of the material presented herein is based on lecture notes of Prof. Suzanne Fielding.

1. Beyond the critical point?

1. Beyond the critical point – Kelvin-Helmholtz



1. Beyond the critical point – Taylor-Couette flow

New base flow: Taylor vortices,

This flow is stable until a second critical rotation rate,

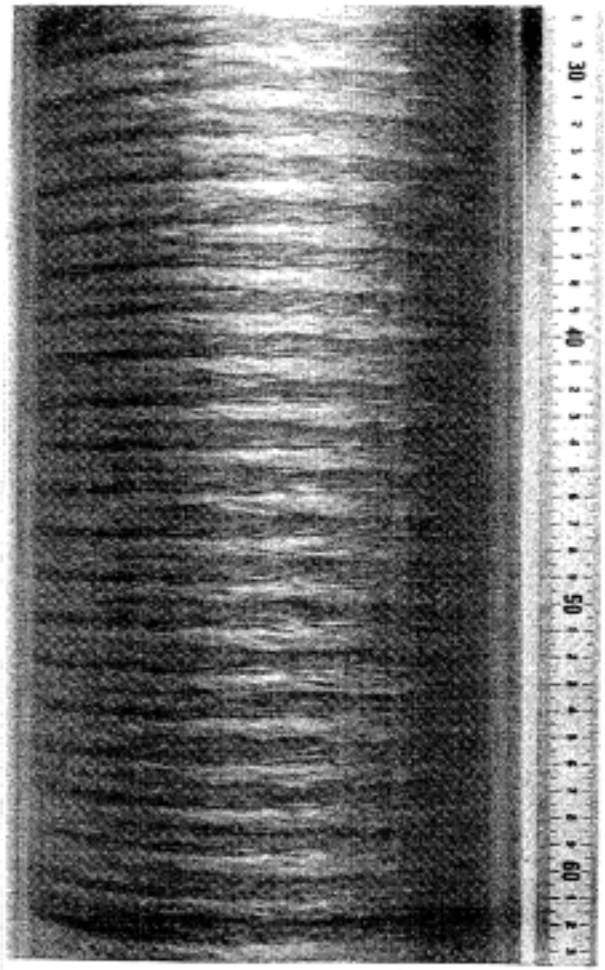
After which a new instability develops

Azimuthal travelling waves.

Further increase in the rotation rate leads to chaotic behaviour:

- irregular time dependence.

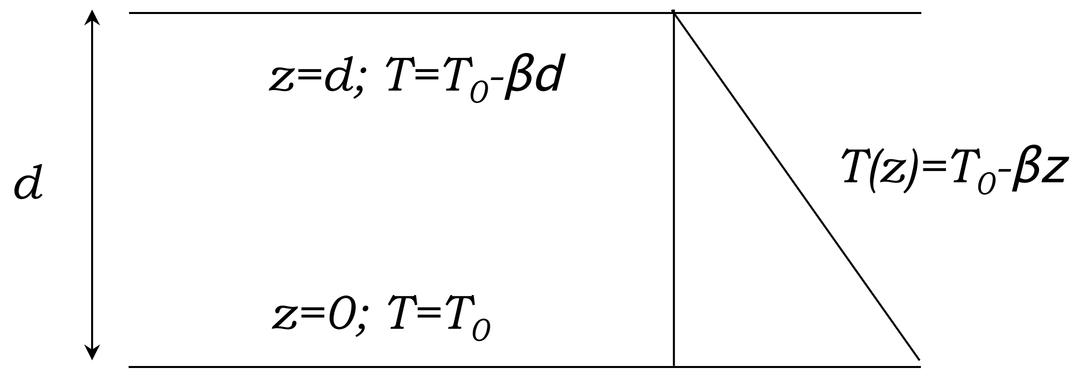
Further rincrease and the flow finally becomes turbulent, but with organised structures still visible…



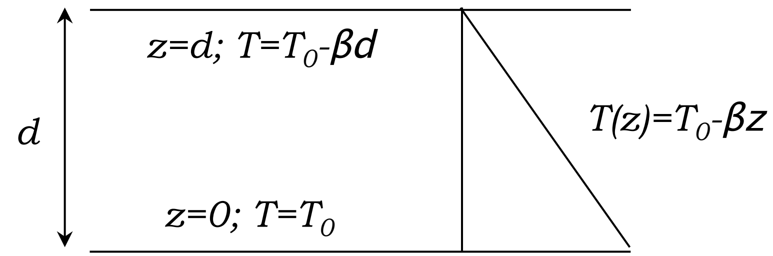
(d) $\Omega_{c4} < \Omega_1$: turbulent Taylor vortices

1. Beyond the critical point – Rayleigh-Bénard convection

Base state: no flow & vertical temperature gradient



Boussinesq approximation



The density changes are associated with changes in volume due to temperature

Coefficient of volume expansion

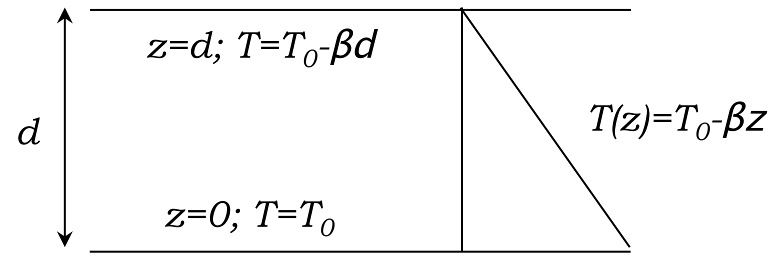
$$\Delta\rho = -\alpha \Delta T \rho_0$$

With which there is an associated body force (buouyancy) proportional to

$$\rho_o \alpha \Delta T g$$

This is what drives the flow, trying to move hot fluid upward.

Boussinesq approximation



Any motion of fluid due to the body force will have to compete with:

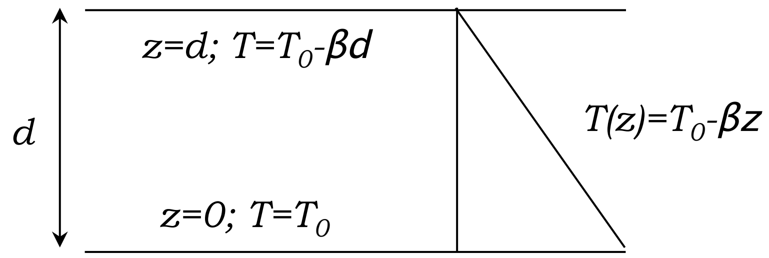
- Viscous diffusion ν
- Thermal diffusivity κ

Prandtl number: $\mathrm{P} = \frac{\nu}{\kappa}$

But the more important non-dimensional number is:

Rayleigh number: $\mathrm{R} = \frac{\alpha \Delta T g d^3}{\nu \kappa}$ *

2. Rayleigh-Bénard convective instability



Prandtl number: $\mathrm{P} = \frac{\nu}{\kappa}$

- Ratio of kinematic viscosity to thermal diffusivity

- Embodies the relative effects of momentum diffusion and heat diffusion

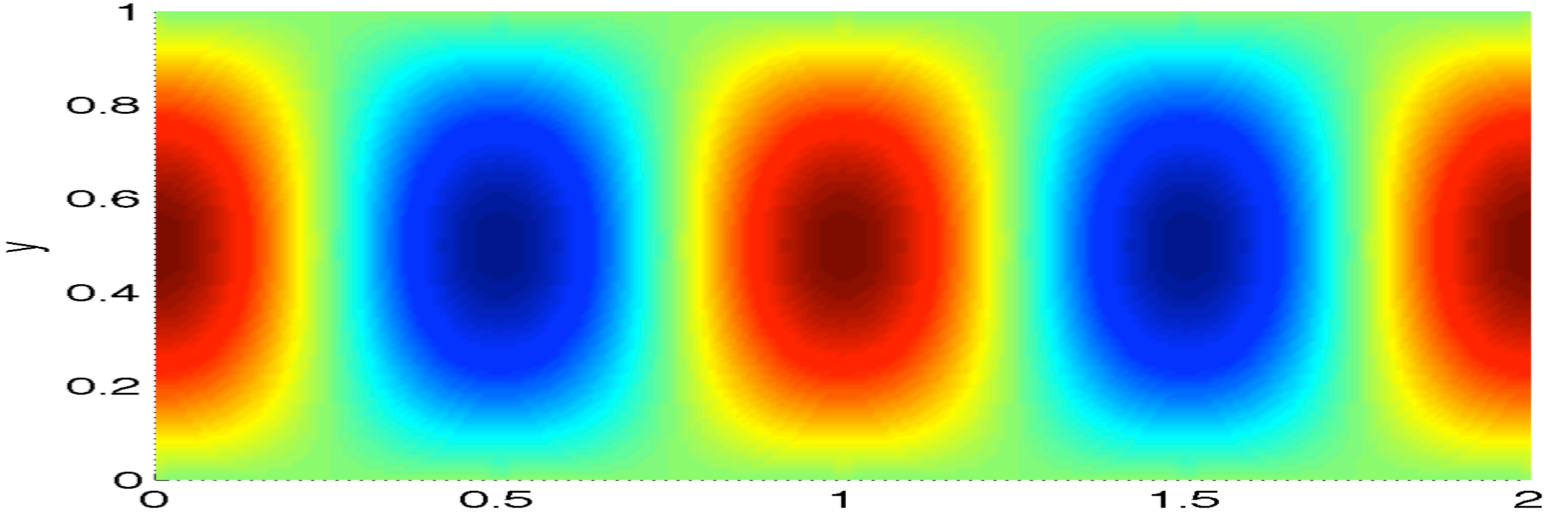
Rayleigh number: $\mathrm{R} = \frac{\alpha \Delta T g d^3}{\nu \kappa}$

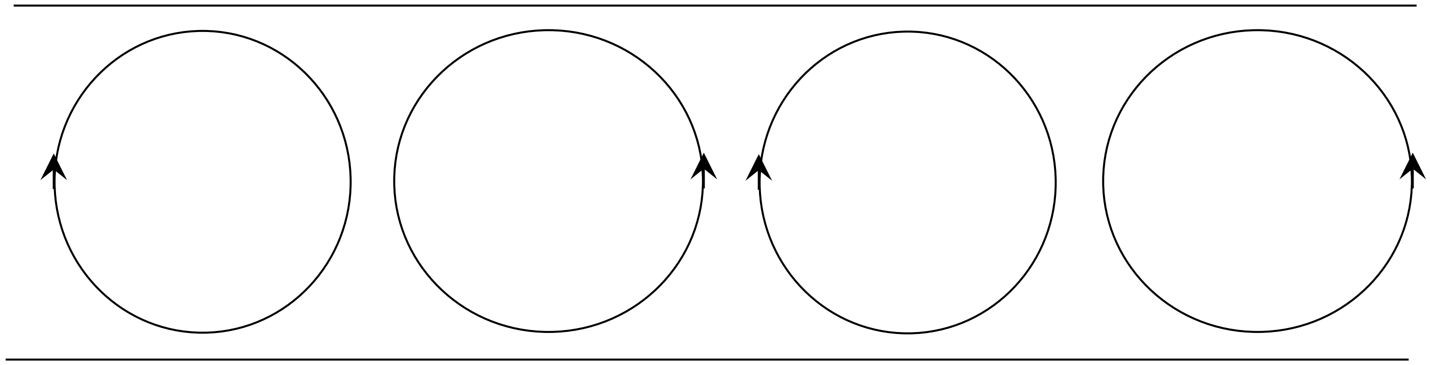
- Ratio of buouyancy effects to the combined viscous and thermal diffusion:

    - Buouyancy is destabilising,
    - Diffusion is stabilising.

- A good stability parameter.

Spatial structure of the instability: $\overline{\omega} = A \sin(\pi z) \cos(a_c x)$





2. Rayleigh-Bénard convective instability

But the flow physics associated with the rolls allow them to persist,

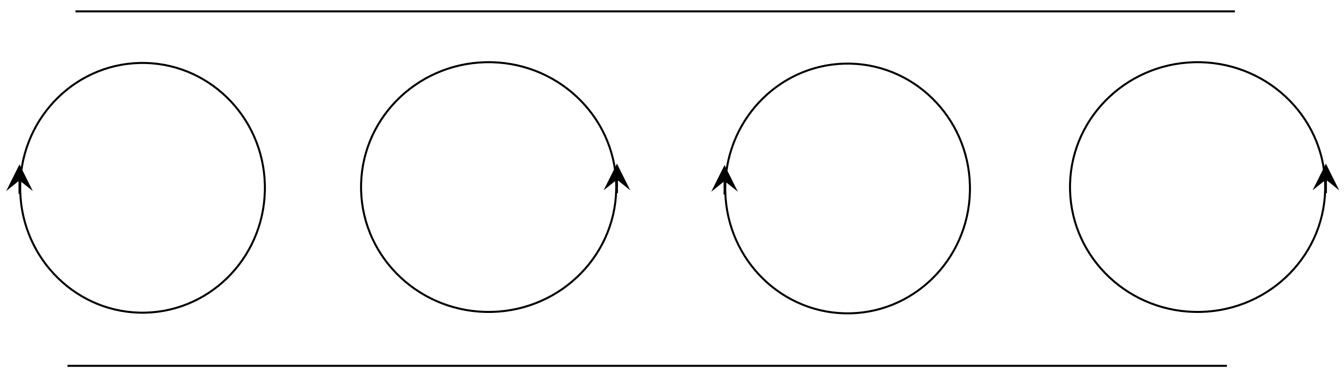
- Heat is transferred away from the lower wall, reducing the potential for instability,

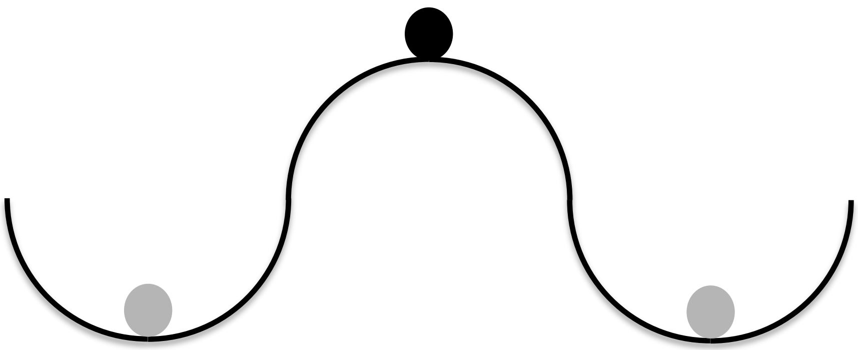
- Horizontal gradients increase viscous friction and thermal diffusion,

The amplitude of the motions saturates beyond threshold:

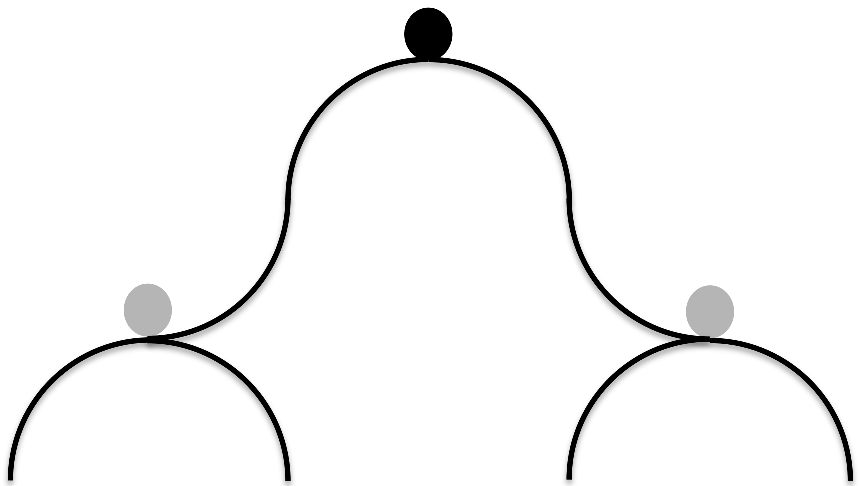
- **bifurcation is *supercritical***, a new base state is established, and this is stable until a new critical value of R is attained,

Numerous bifurcations can occur before the flow reaches a turbulent state.



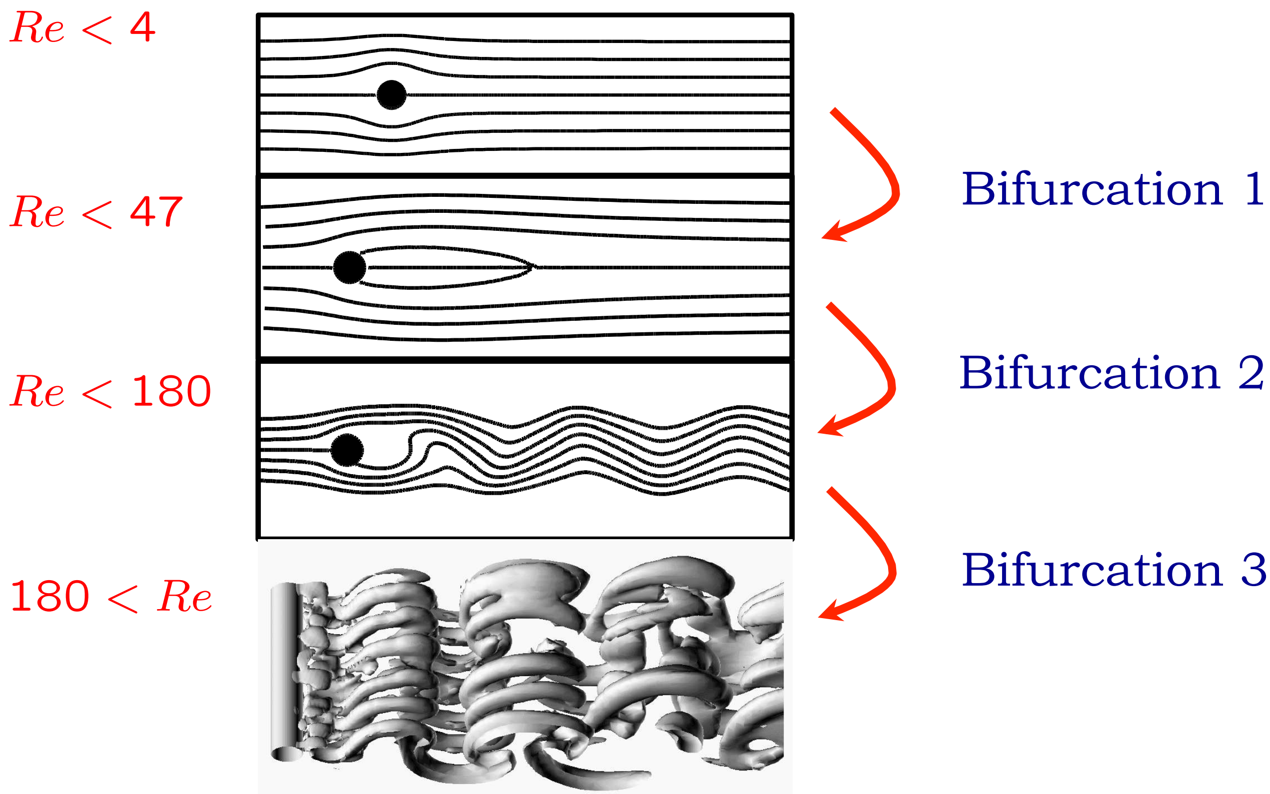


Non-linearity will sometimes stabilise the system at a new base state, which persists until a second critical point is reached.



Non-linearity will sometimes stabilise the system at a new base state, which persists until a second critical point is reached.

1. Beyond the critical point – cylinder wake



Noack & Ecklemann JFM 1994

i. Kelvin-Helmholtz: direct transition to turbulence,

ii. Taylor-Couette flow, Rayleigh-Bénard convection, cylinder wake:

- series of **bifurcations** and base-flows

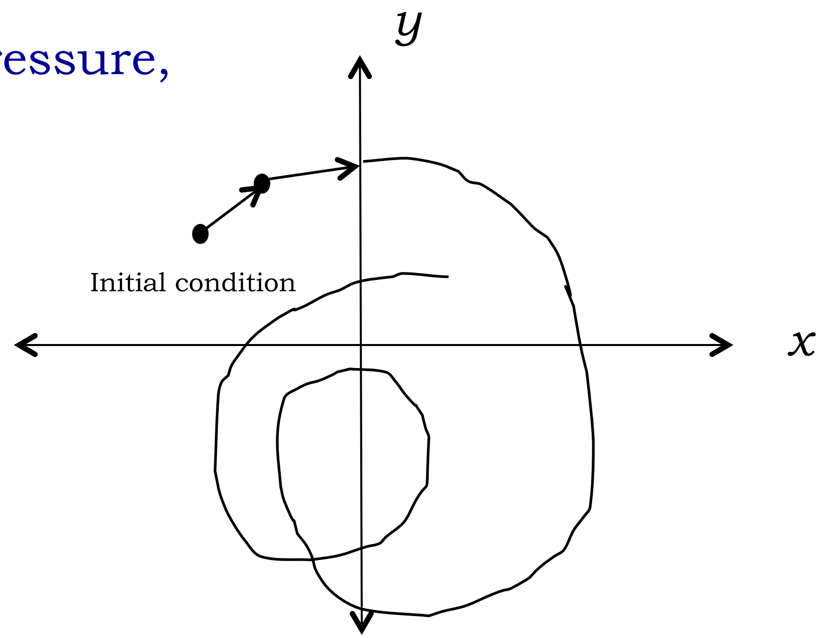
2. State-space representation of dynamical systems

Dynamics viewed in phase or state space

- state variables: minimum number of variables that uniquely define the status of the system,

- could be velocities, momenta, pressure,
- voltage, current,
- heat flux,...

$$\frac{dx}{dt} = f_1(x, y)$$
$$\frac{dy}{dt} = f_2(x, y)$$



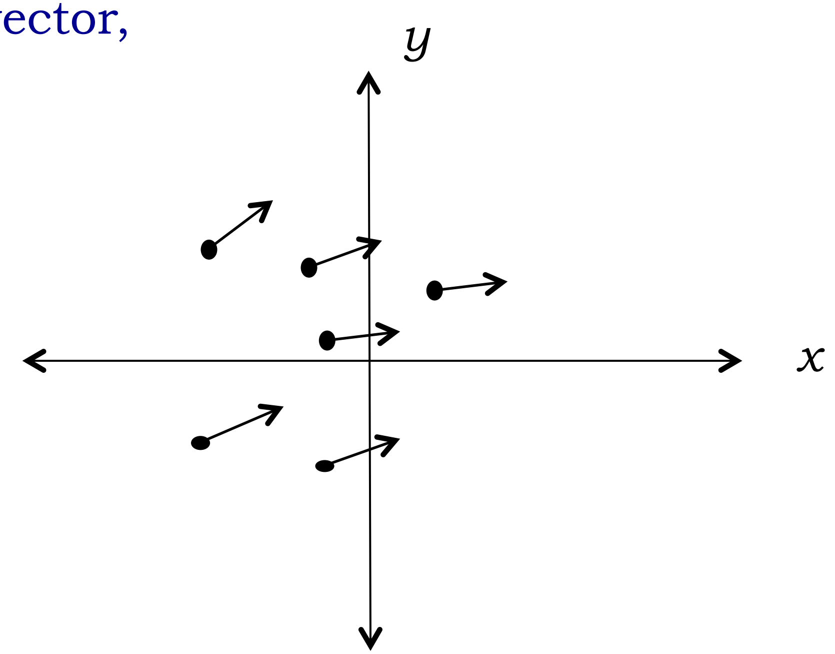
*

A system of non-linear differential equations represents a flow in a state space: incrementally this is a series of vectors.

Dynamics viewed in phase or state space

- for each initial condition there is a different trajectory,
- trajectory is the solution for a given IC,
- at each point the DEs define a vector,

$$\frac{dx}{dt} = f_1(x, y)$$
$$\frac{dy}{dt} = f_2(x, y)$$

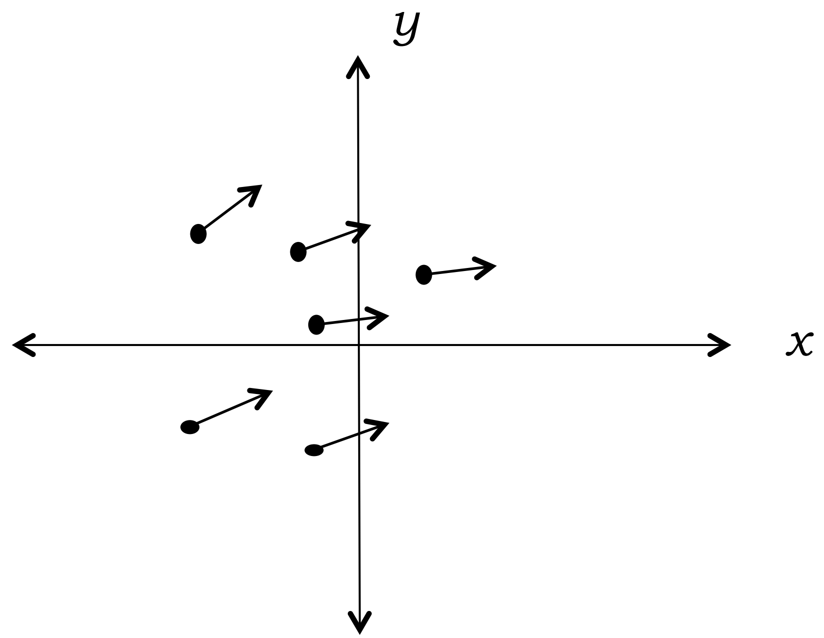


Dynamics viewed in phase or state space

- To know all of the possible dynamics one would need to solve the system for every initial condition,

- clearly not feasible !

$$\frac{dx}{dt} = f_1(x, y)$$
$$\frac{dy}{dt} = f_2(x, y)$$



But we can obtain a qualitative, graphical, representation of the state-space and the vector field (flow), and this can sometimes provide access to the qualitative behaviour of the non-linear dynamics.

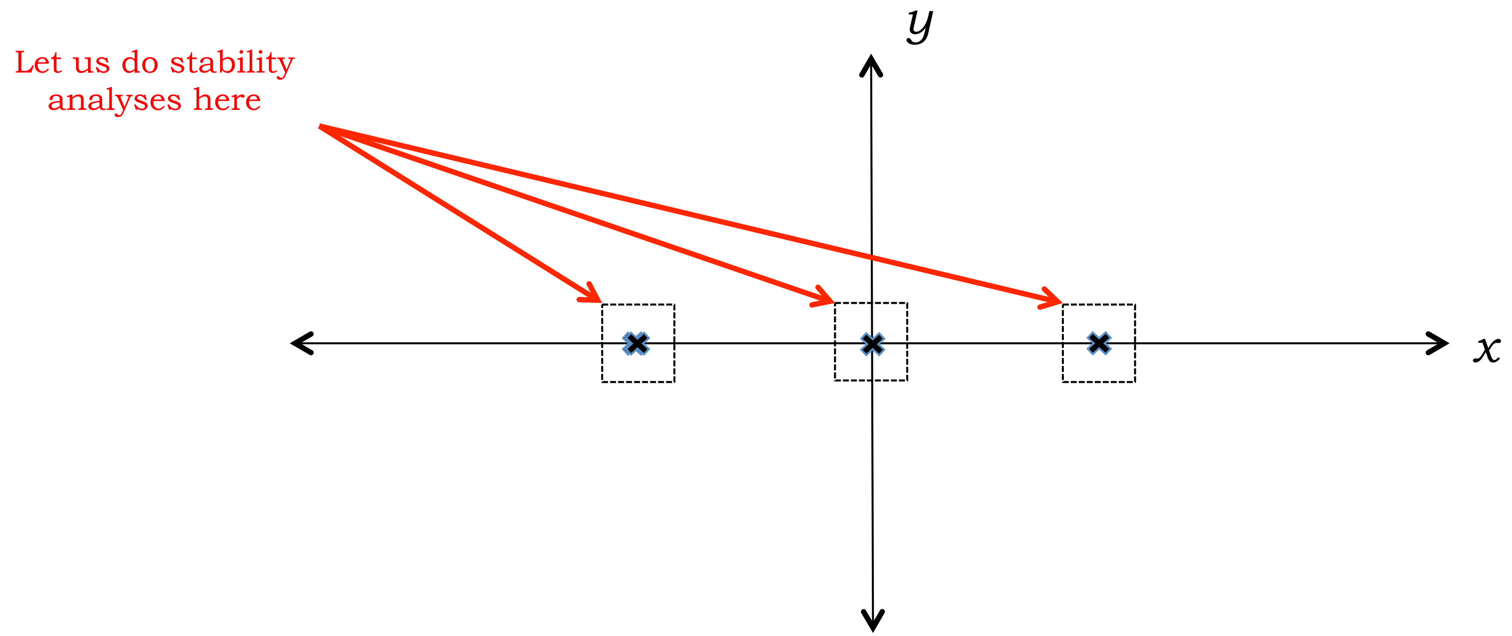
Dynamical system →

$$\dot{x} = y$$
$$\dot{y} = -x + x^3$$

Fixed points $(-1, 0), (0, 0), (+1, 0)$

The base flows you've been using for stability analysis are all fixed points

What is the character of the vector field in the vicinity of these points?



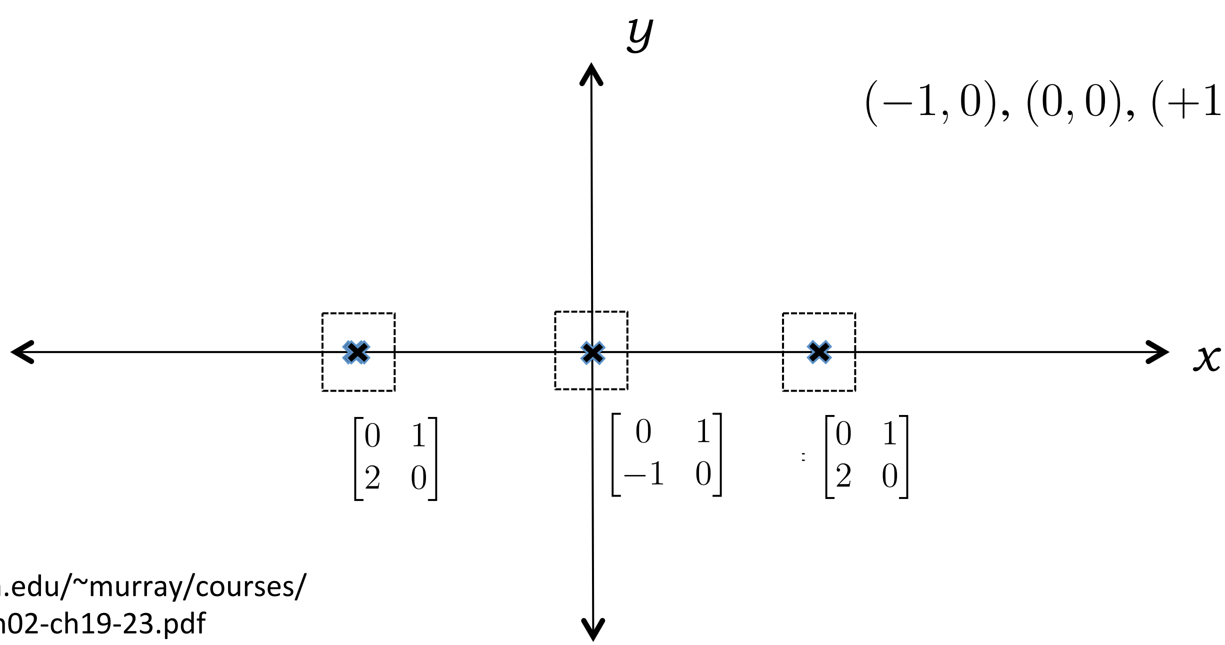
3. State-space representation of dynamical systems

What is the character of the vector field in the vicinity of these points?

Zoom in and use Jacobian to obtain local linearisation*

$$\begin{bmatrix} \delta\dot{x} \\ \delta\dot{y} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} \longrightarrow \begin{bmatrix} \delta\dot{x} \\ \delta\dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3x^2 - 1 & 0 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$$

$(-1, 0), (0, 0), (+1, 0)$ *

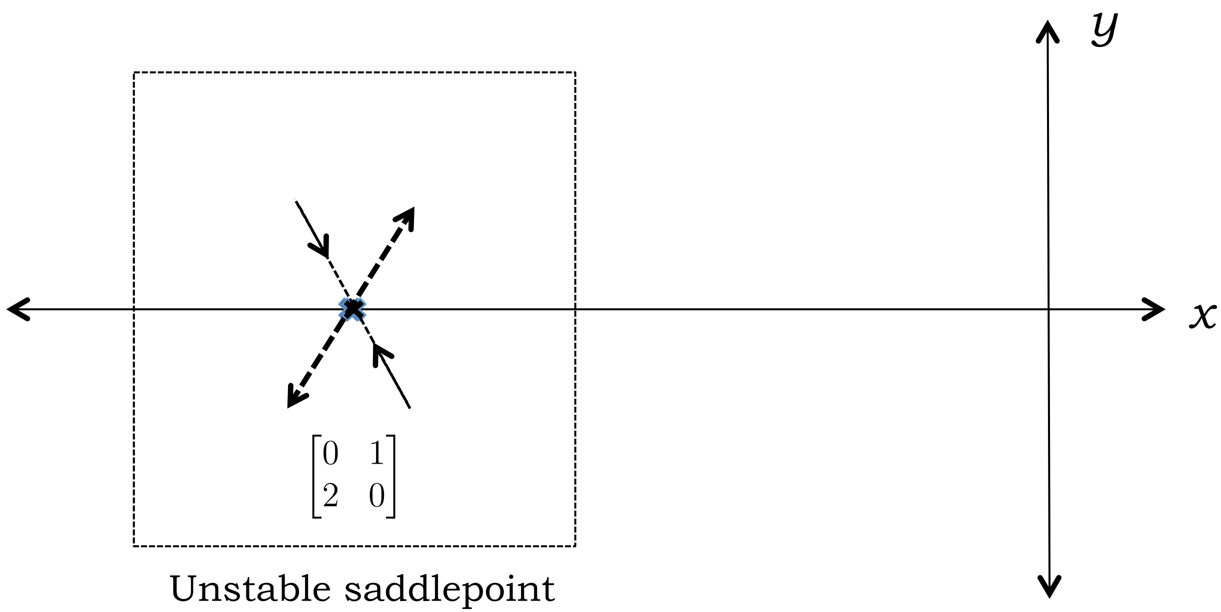


*http://www.cds.caltech.edu/~murray/courses/cds101/fa02/caltech/pph02-ch19-23.pdf

What is the character of the vector field in the vicinity of these points?

Compute the eigenvalues and eigenvectors

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \xrightarrow{\det[A-\lambda I]=0} \quad \lambda^2 = 2, \quad \lambda = \pm\sqrt{2} \qquad \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix} \quad \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}$$

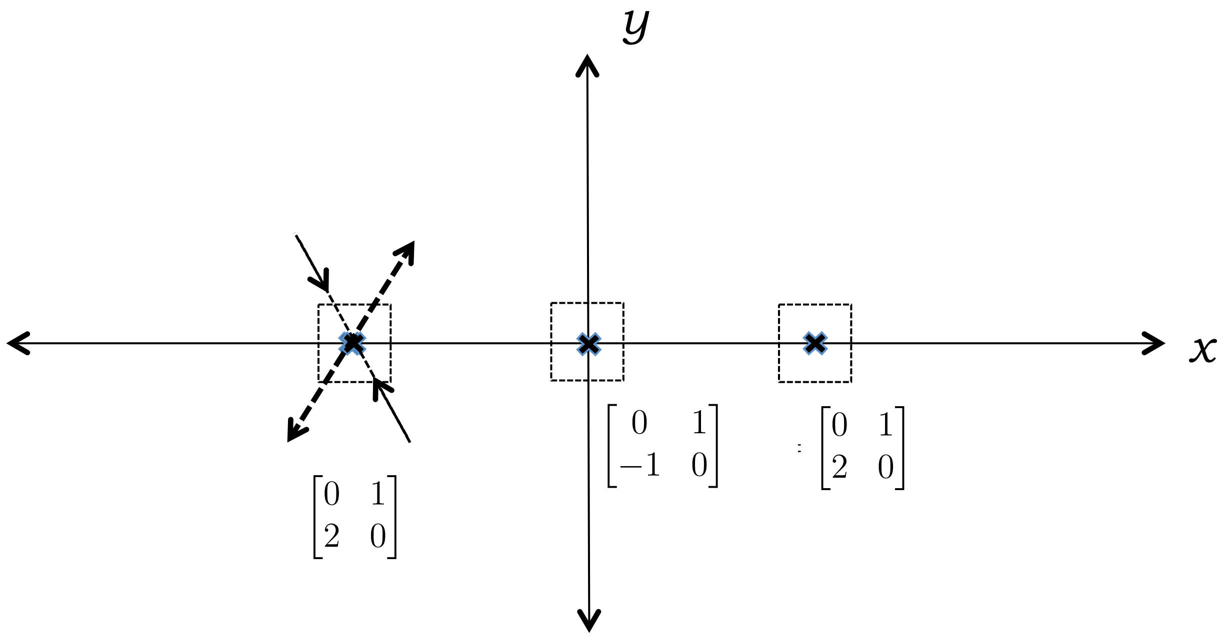


Unstable saddlepoint

What is the character of the vector field in the vicinity of these points?

Zoom in and use Jacobian to obtain local linearisation

$$\begin{bmatrix} \delta\dot{x} \\ \delta\dot{y} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} \longrightarrow \begin{bmatrix} \delta\dot{x} \\ \delta\dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3x^2 - 1 & 0 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$$



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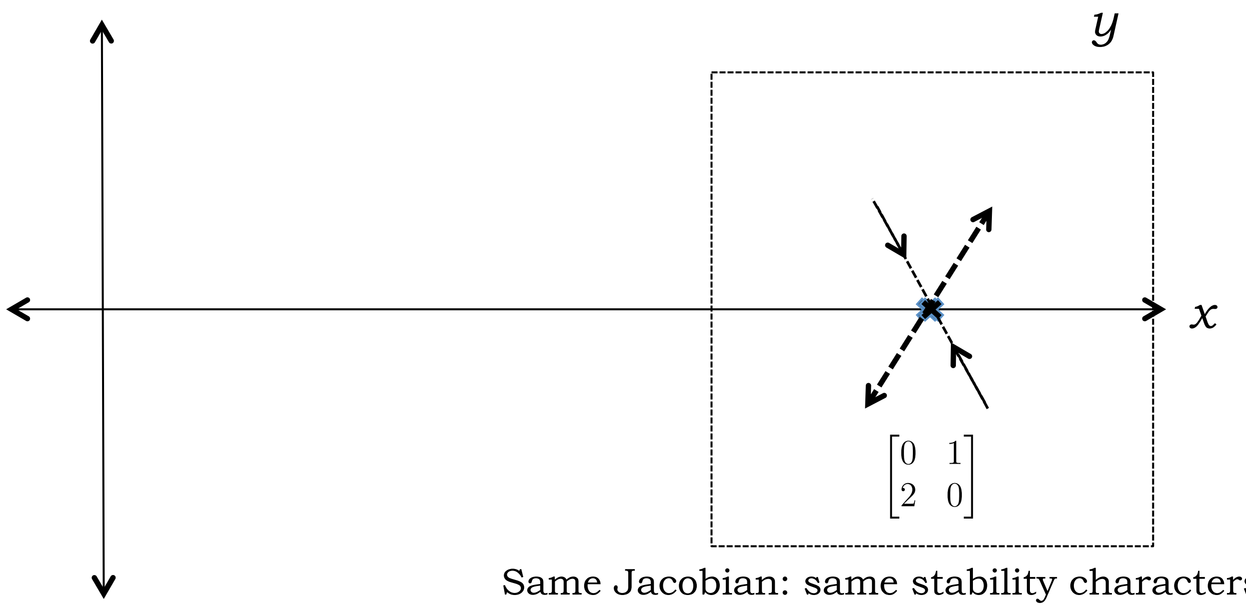
Compute the eigenvalues and eigenvectors

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \xrightarrow{\det[A-\lambda I]=0} \quad \lambda^2 = 2 \quad \lambda = \pm\sqrt{2} \qquad \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix} \quad \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}$$



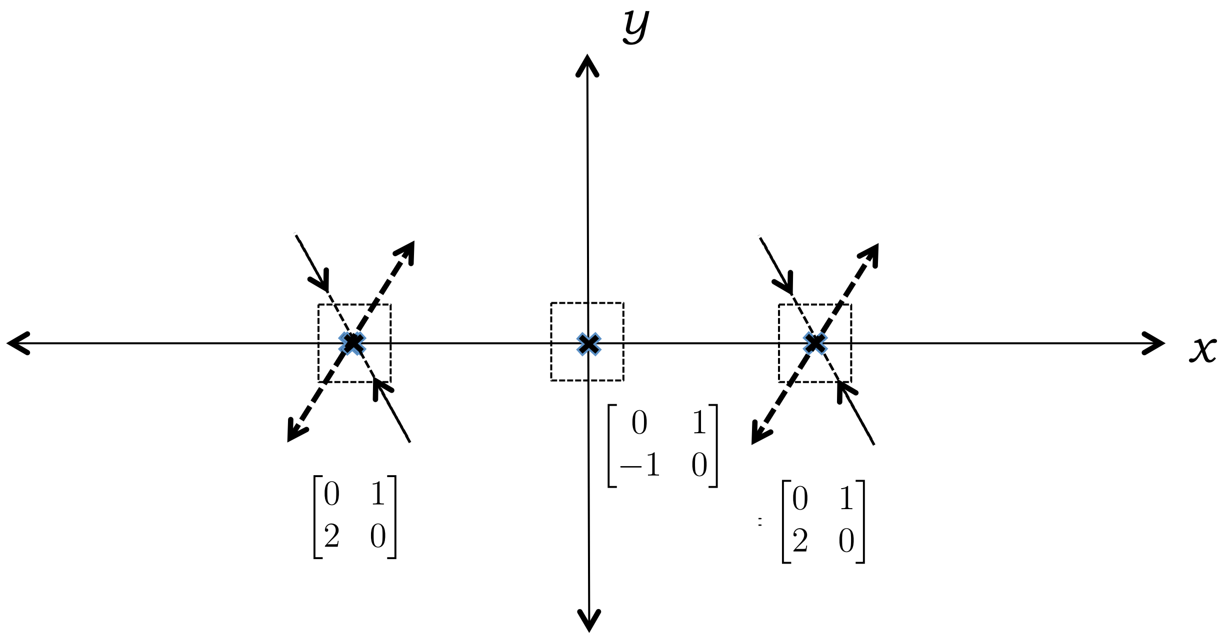
$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

Same Jacobian: same stability charactersitics

What is the character of the vector field in the vicinity of these points?

Zoom in and use Jacobian to obtain local linearisation

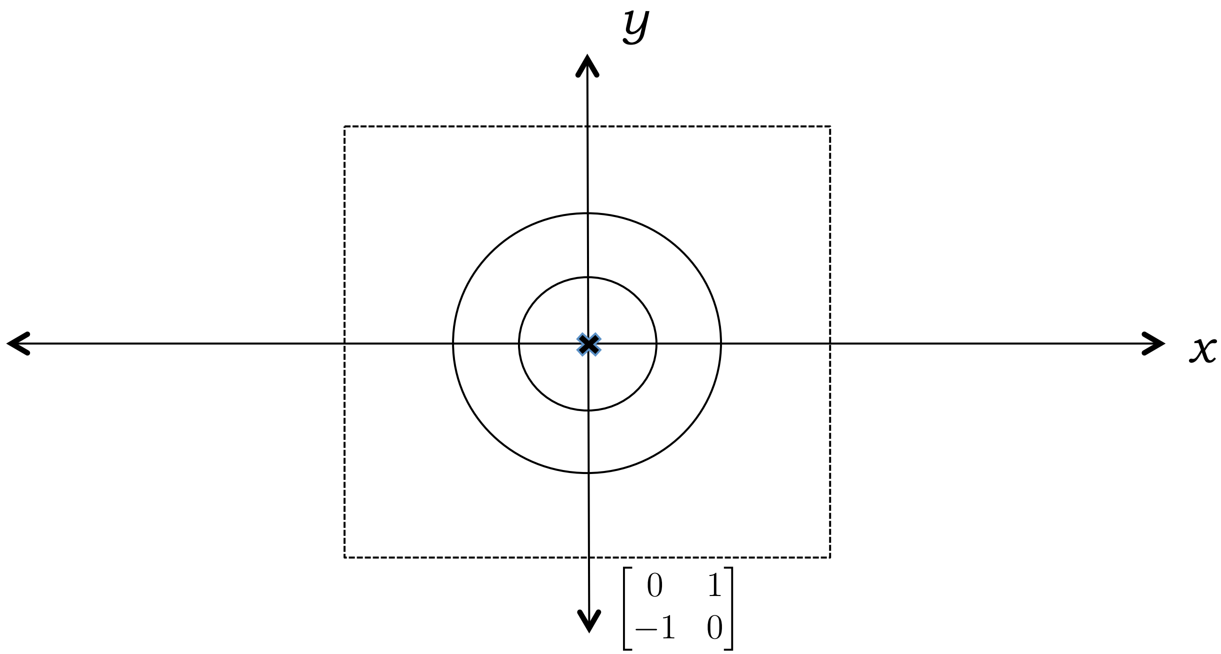
$$\begin{bmatrix}\delta\dot{x}\\ \delta\dot{y}\end{bmatrix} = \begin{bmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{bmatrix}\begin{bmatrix}\delta x\\ \delta y\end{bmatrix} \longrightarrow \begin{bmatrix}\delta\dot{x}\\ \delta\dot{y}\end{bmatrix} = \begin{bmatrix}0 & 1\\ 3x^2-1 & 0\end{bmatrix}\begin{bmatrix}\delta x\\ \delta y\end{bmatrix}$$



What is the character of the vector field in the vicinity of these points?

Zoom in and use Jacobian to obtain local linearisation

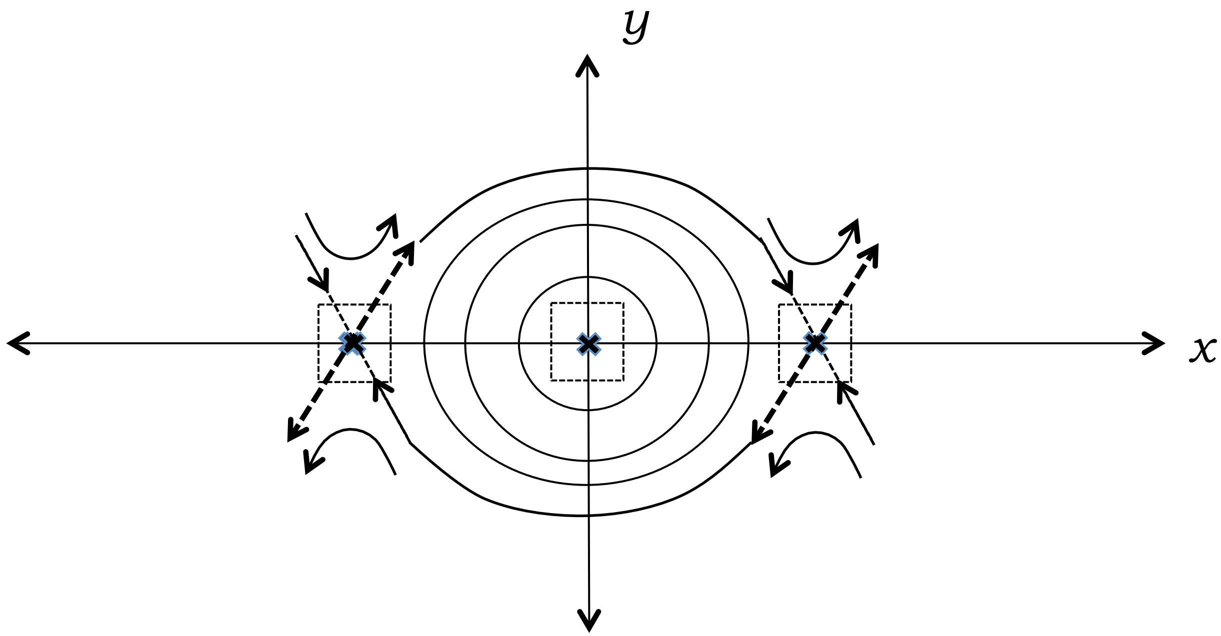
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \xrightarrow{\det[A - \lambda I] = 0} \begin{aligned} \lambda^2 &= 1 \\ \lambda &= \pm i \end{aligned}$$

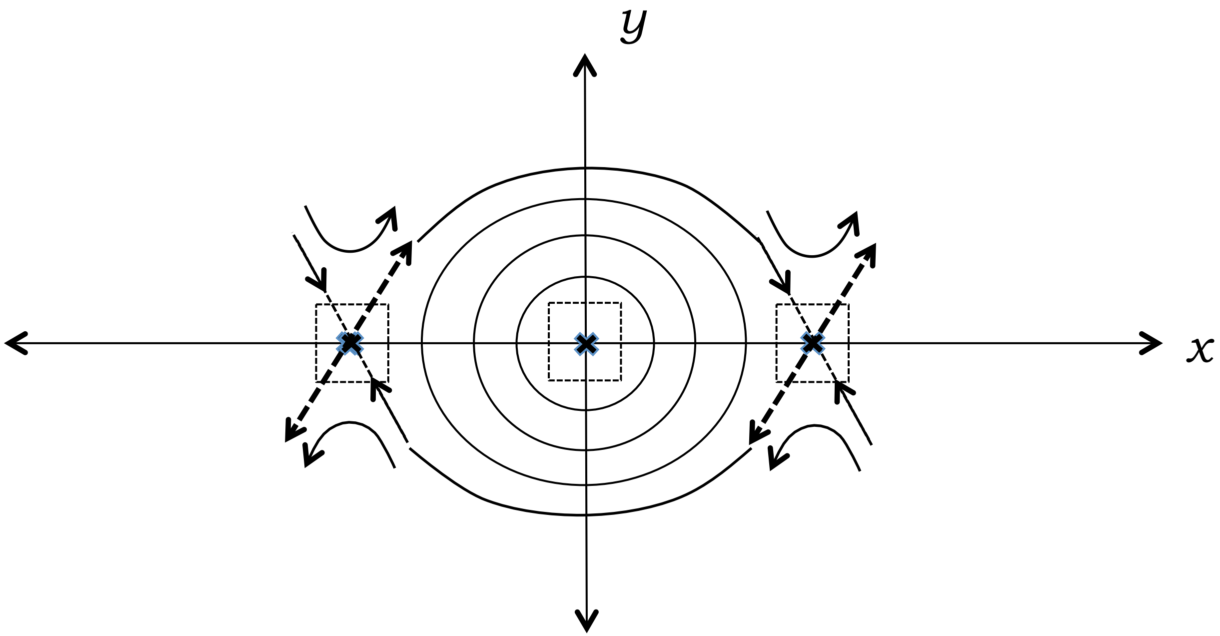


$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Once the stability characteristics have been determined in the vicinity of the fixed points, because **solution trajectories can never cross paths**, and are smooth, we can complete the **qualitative solution topology** in regions where non-linear dynamics will be manifest.

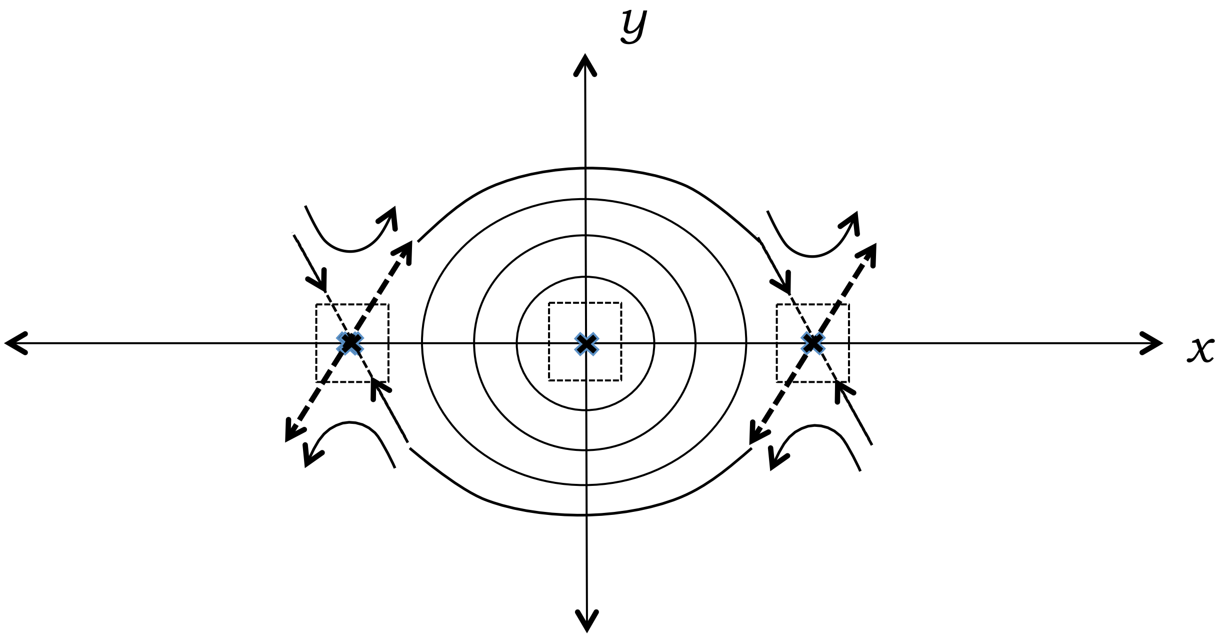
$$\dot{x} = y$$
$$\dot{y} = -x + x^3$$





The continuation of an **eigenvector** into the non-linear regime is known as a **manifold**, which can be also be classified as stable or unstable,

Unstable eigenvectors are tangent to unstable manifolds at fixed points,



In systems whose dynamics depend on some parameter, the vector fields change as we go through a **bifurcation point (e.g critical Re or R)**:

- the stability characteristics of one or more of the fixed points change…

3. Local bifurcation theory

We will use simple model equations to explore some of the more frequently encountered bifurcation scenarios:

- Saddlenode bifurcation,
- Transcritical bifurcation,
- Pitchfork bifurcation,
- Hopf bifurctaion,
- Bifurcation in the Lorenz system.

The saddlenode bifurcation

$$\frac{dx}{dt} = a - x^2 \quad \text{for} \quad x, a \text{ real.}$$

What are the base states?

*

$$\frac{dx}{dt} = 0 \qquad\qquad x = x_B = \pm\sqrt{a}$$

$$\frac{dx}{dt} = a - x^2 \quad \text{for} \quad x, a \ \text{real.}$$

Possibilities:

$$x = x_B = \pm\sqrt{a}$$

*

$a < 0$ no real solution,

$a > 0$ two real solutions.

Assess the stability of each of the base states

$$\frac{dx}{dt} = a - x^2 \qquad x = x_B = \pm\sqrt{a} \qquad a > 0$$

Add a perturbation and subtitute into governing equation *

$$x = x_B + \tilde{x}$$

0 Linearisation: negligible

$$\frac{d\tilde{x}}{dt} = (a - x_B^2) - 2x_B\tilde{x} - \tilde{x}^2$$

$$\frac{d\tilde{x}}{dt} = -2x_B\tilde{x}$$

$$\frac{d\tilde{x}}{dt} = -2x_B\tilde{x}$$

Write down the solution

$$\tilde{x}(t) = A\mathrm{e}^{-2x_B t}$$

*

This gives us the stability of the two base states

$$\text{for } x_B = +\sqrt{a},\ |\tilde{x}| \to 0 \text{ as } t \to \infty$$

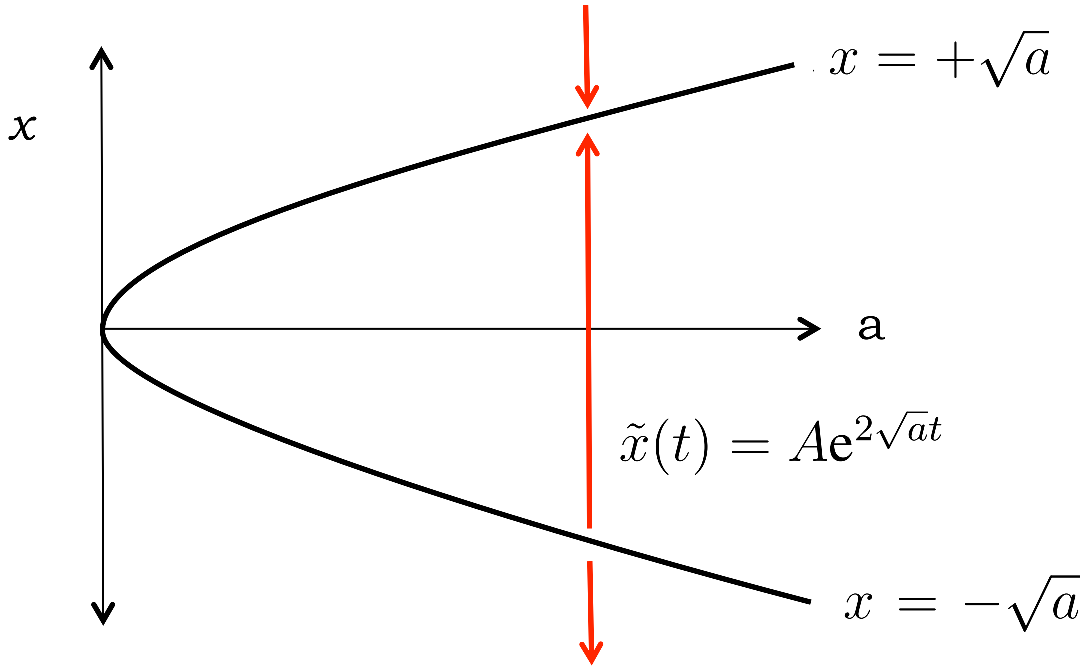
$$\text{for } x_B = -\sqrt{a},\ |\tilde{x}| \to \infty \text{ as } t \to \infty$$

The saddlenode bifurcation

$$\text{for } x_B = +\sqrt{a},\ |\tilde{x}| \to 0 \text{ as } t \to \infty$$

$$\text{for } x_B = -\sqrt{a},\ |\tilde{x}| \to \infty \text{ as } t \to \infty$$

a=0 is a bifurcation point



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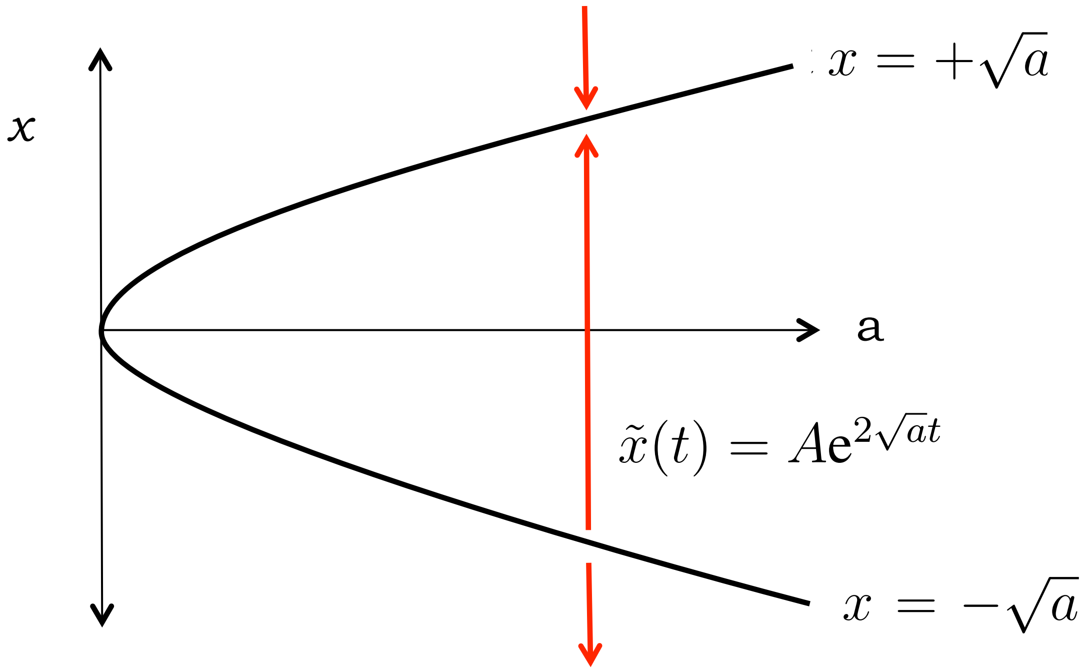
The saddlenode bifurcation



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The bifurcation point, *a=0*, corresponds to the appearance of two new **solution branches**, one stable, the other ustable,

The solution branches are due to the non-linearity, just like in the Rayleigh-Bénard and cylinder-wake examples.

The transcritical bifurcation

Dynamics governed by a system with two control parameters

$$\frac{dx}{dt} = ax - bx^2 \quad \text{for} \quad x, a, b \text{ real.}$$

Write down the steady states

$$x = x_{B1} = 0$$

$$x = x_{B2} = a/b$$

$$\frac{dx}{dt} = ax - bx^2 \quad \text{for} \quad x, a, b \text{ real.}$$

$$x = x_{B1} = 0$$

$$x = x_{B1} + \tilde{x}$$

Write down the linearised system for x_{B1}

$$\frac{d\tilde{x}}{dt} = a\tilde{x} - b\tilde{x}^2$$

$$\frac{d\tilde{x}}{dt} = a\tilde{x}$$

$$x = x_{B1} = 0 \qquad \frac{d\tilde{x}}{dt} = a\tilde{x}$$

Write down the solution

$$\tilde{x}(t) = A\mathrm{e}^{at}$$

Stability criteria?

$a < 0$ Linearly stable

$a > 0$ Linearly unstable

$$\frac{dx}{dt} = ax - bx^2 \quad \text{for} \quad x, a, b \text{ real.}$$

$$x = x_{B2} = a/b$$

*

Write down the linearised system and solution for x_{B2}

$$\frac{d\tilde{x}}{dt} = -a\tilde{x} \qquad \tilde{x}(t) = A\mathrm{e}^{-at}$$

$$\frac{d\tilde{x}}{dt} = -a\tilde{x} \qquad \tilde{x}(t) = A\mathrm{e}^{-at}$$

Stability criteria?

$a < 0$ Linearly unstable

$a > 0$ Linearly stable

The transcritical bifurcation

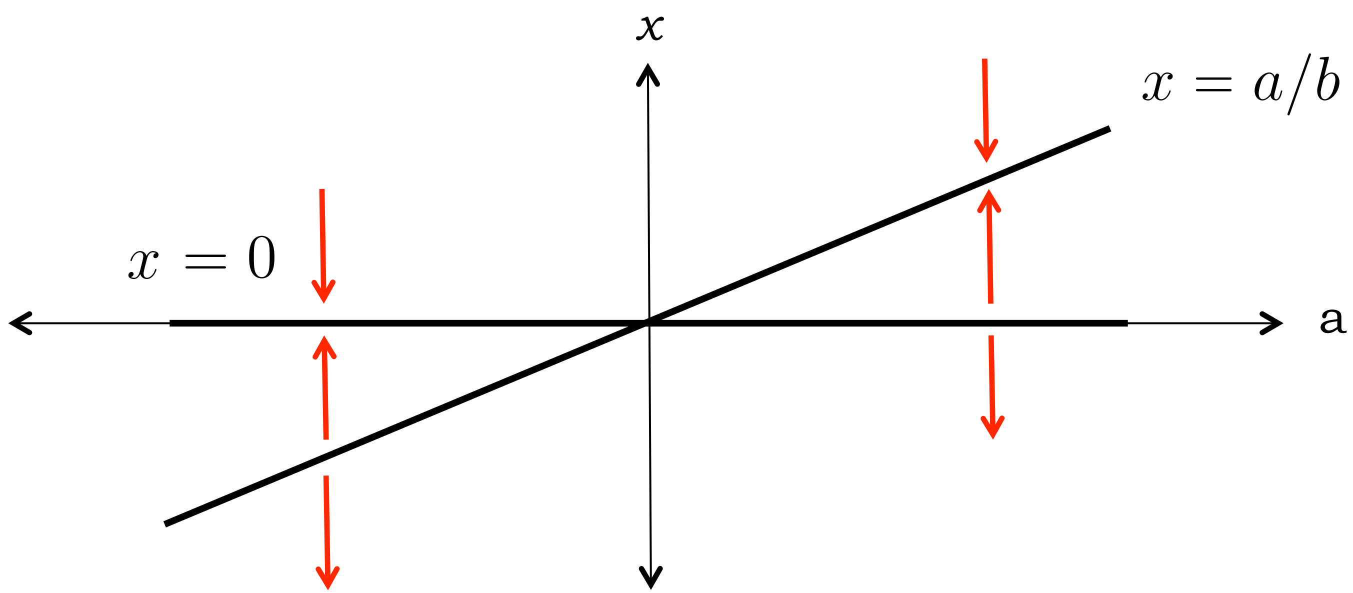
$$x = x_{B1} = 0$$

$$\tilde{x}(t) = A\mathrm{e}^{at}$$

$$x = x_{B2} = a/b$$

$$\tilde{x}(t) = A\mathrm{e}^{-at}$$

	$x = x_{B1} = 0$	$x = x_{B2} = a/b$
$a < 0$	Linearly **stable**	Linearly **unstable**
$a > 0$	Linearly **unstable**	Linearly **stable**



The pitchfork bifurcation

Dynamics governed by a system with two control parameters

$$\frac{dx}{dt} = ax - bx^3 \quad \text{for} \quad x, a, b \ \text{real}$$

Steady states

$$x = x_{B1} = 0$$
$$x = x_{B2} = +\sqrt{a/b} \quad \text{for} \quad a/b > 0$$
$$x = x_{B3} = -\sqrt{a/b} \quad \text{for} \quad a/b > 0$$

Base states 2 and 3 only exist for *a>0* if *b>0.*

Linear stability of $x = x_{B1} = 0$

$$\frac{dx}{dt} = ax - bx^3$$

$$x = x_{B1} + \tilde{x} \longrightarrow \frac{d\tilde{x}}{dt} = a\tilde{x}$$

Solution $\tilde{x} = A\mathrm{e}^{at}$

Stability $a < 0$ Linearly stable

$a > 0$ Linearly unstable

Linear stability of

$$x = x_{B2} = +\sqrt{a/b}$$
$$x = x_{B3} = -\sqrt{a/b}$$

$$x = \pm\sqrt{a/b} + \tilde{x}$$

$$\frac{dx}{dt} = ax - bx^3 \longrightarrow \frac{d\tilde{x}}{dt} = a\tilde{x} - 3bx_B^2\tilde{x}$$ *

General solution $\tilde{x} = A\mathrm{e}^{st}$

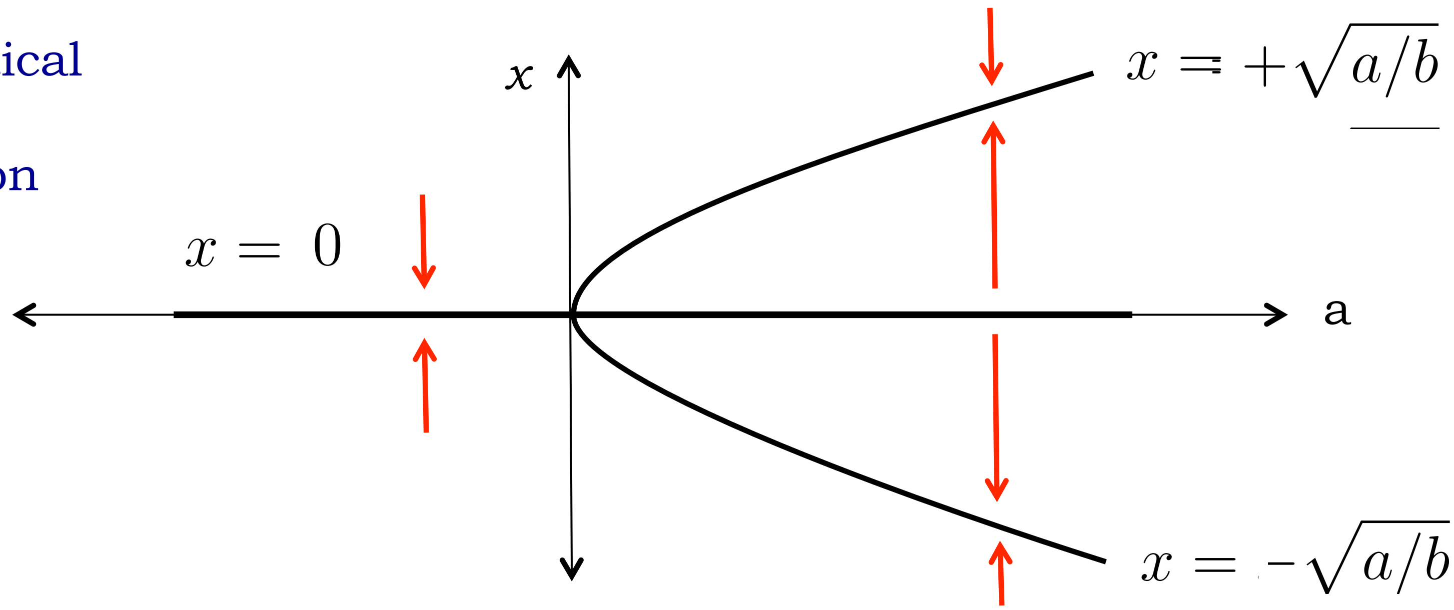
$$s = a - 3bx_B^2 = a - 3b\frac{a}{b} = -2a$$

$$x = x_{B1} = 0$$

$$x = x_{B2} = +\sqrt{a/b}$$
$$x = x_{B3} = -\sqrt{a/b}$$

$a < 0$	Linearly stable	
$a > 0$	Linearly unstable	Linearly stable

Supercritical Pitchfork bifurcation



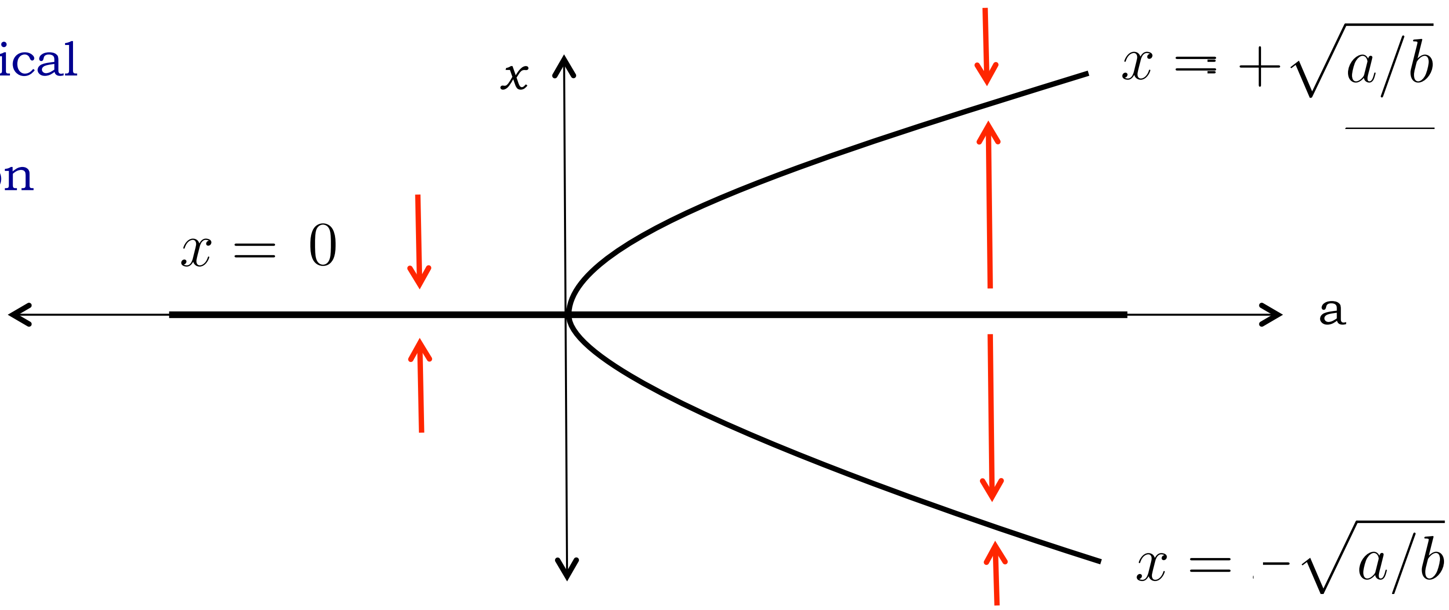
$$x = x_{B1} = 0$$

$$x = x_{B2} = +\sqrt{a/b}$$
$$x = x_{B3} = -\sqrt{a/b}$$

$a < 0$ Linearly stable

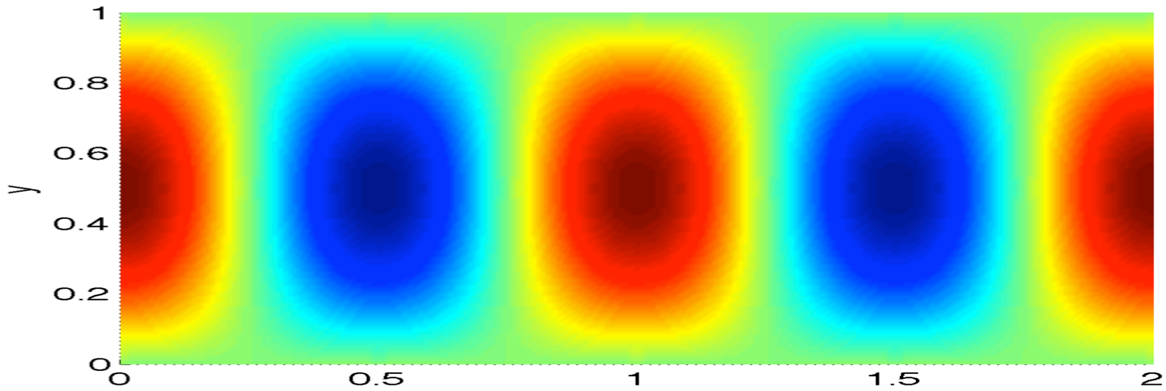
$a > 0$ Linearly unstable Linearly stable

Supercritical Pitchfork bifurcation

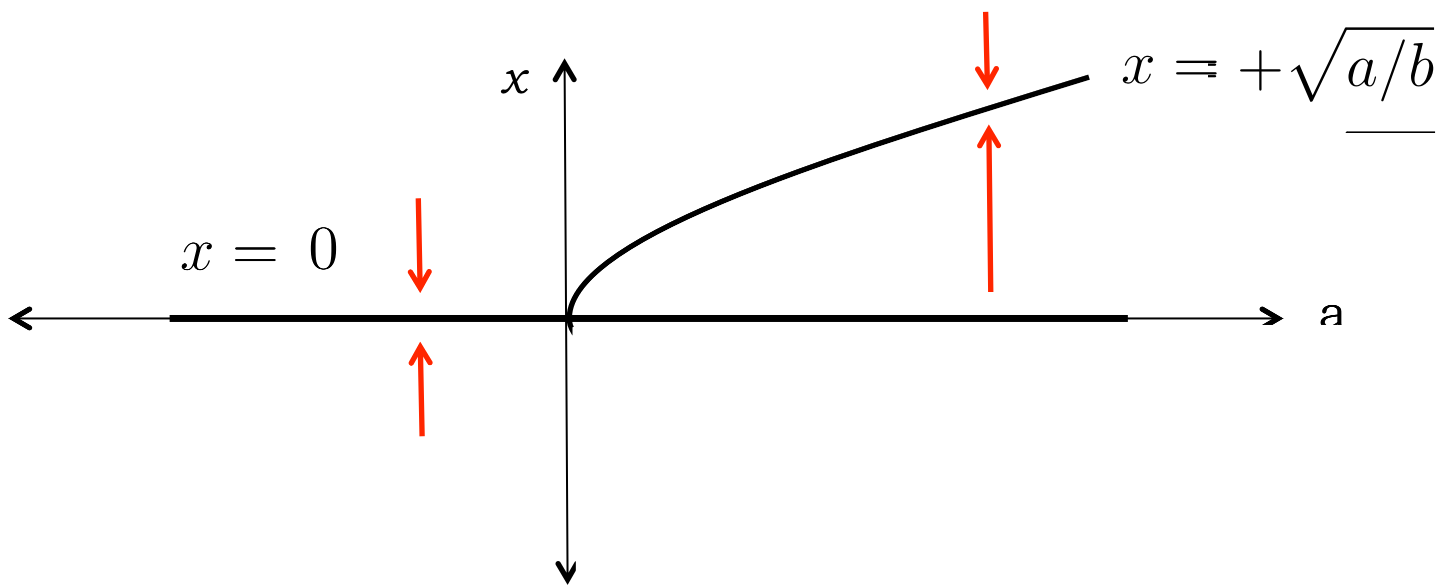


The pitchfork bifurcation

Rayleigh-Bénard is analogous to this

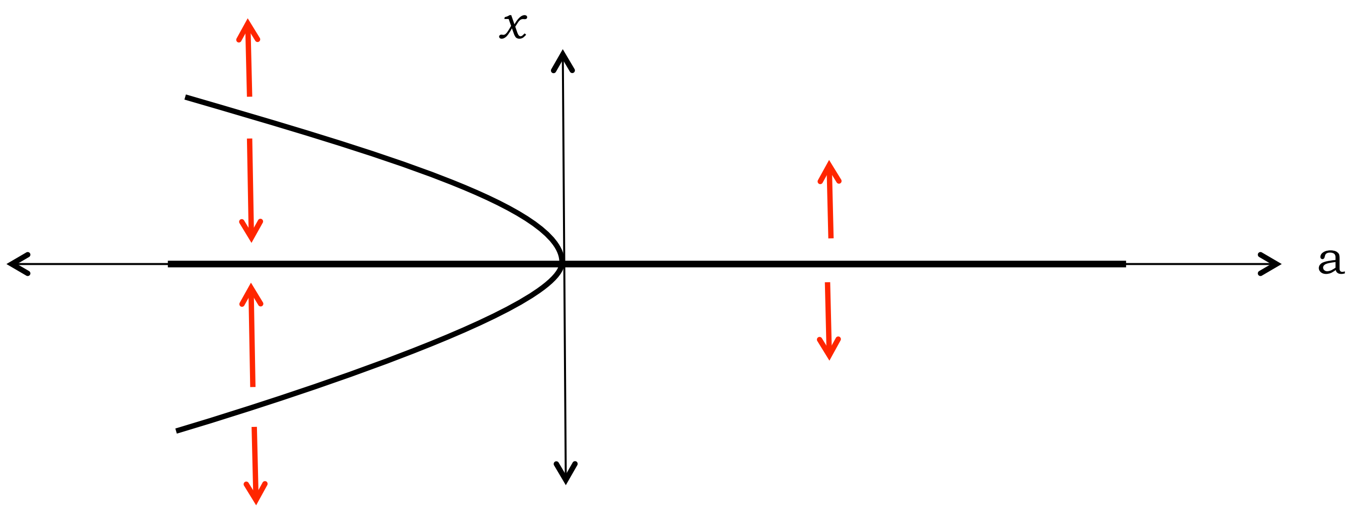


Supercritical Pitchfork bifurcation

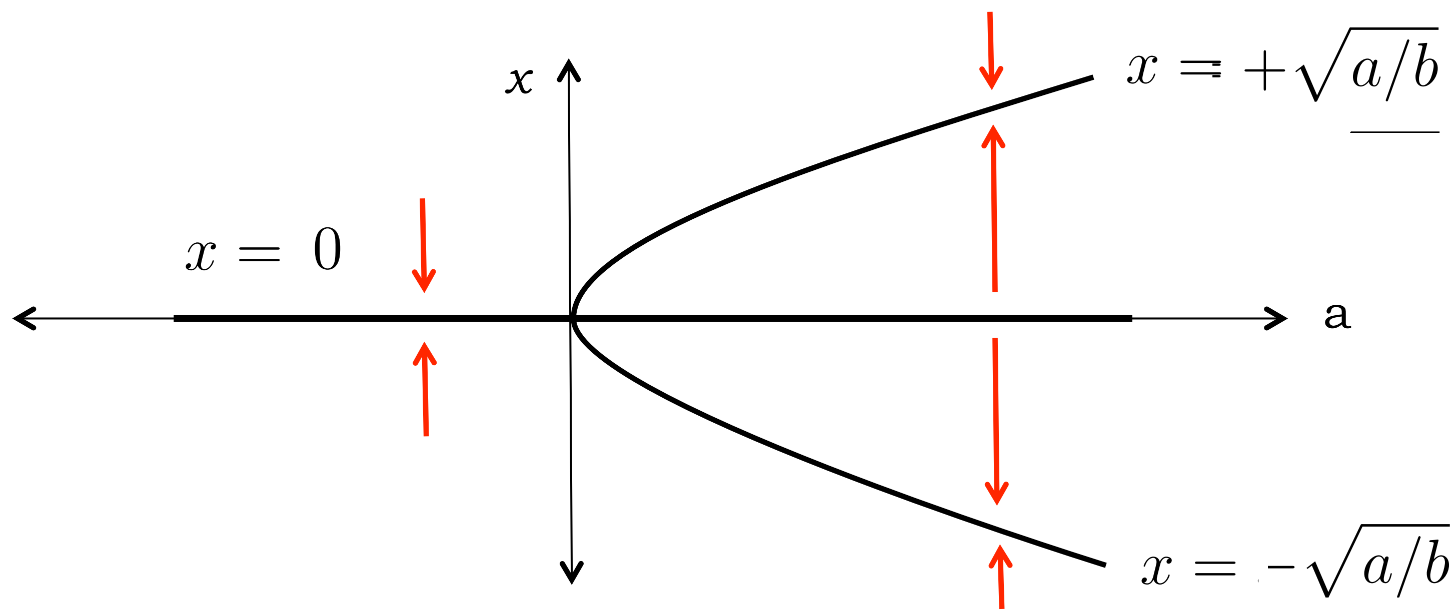


The pitchfork bifurcation

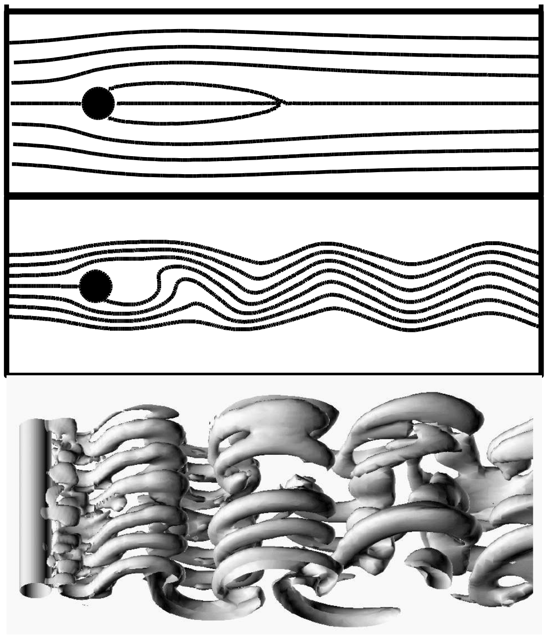
Subcritical Pitchfork bifurcation



Supercritical Pitchfork bifurcation



The Hopf bifurcation



Dynamics governed by two equations

$$\frac{dx}{dt} = -y + (a - x^2 - y^2)x$$
$$\frac{dy}{dt} = x + (a - x^2 - y^2)y$$

Steady state : $x = y = 0$

Base + perturbation

$$x = 0 + \tilde{x}$$
$$y = 0 + \tilde{y}$$

What is the linearised system?

$$\frac{d\tilde{x}}{dt} = -\tilde{y} + a\tilde{x}$$
$$\frac{d\tilde{y}}{dt} = \tilde{x} + a\tilde{y}$$

$$\frac{d\tilde{x}}{dt} = -\tilde{y} + a\tilde{x}$$
$$\frac{d\tilde{y}}{dt} = \tilde{x} + a\tilde{y}$$

Solution (normal modes)

$$\tilde{x} = \alpha \mathrm{e}^{st} + \mathrm{c.c.}$$
$$\tilde{y} = \beta \mathrm{e}^{st} + \mathrm{c.c.}$$

*

Substituting into linearised system

$$\alpha s = -\beta + a\alpha$$
$$\beta s = \alpha + a\beta$$

$$s^2 - 2as + (a^2 + 1) = 0 \longrightarrow s = a \pm i$$

The Hopf bifurcation

$$\tilde{x} = \alpha \mathrm{e}^{st} + \text{c.c.}$$
$$\tilde{y} = \beta \mathrm{e}^{st} + \text{c.c.}$$

$$s = a \pm i$$

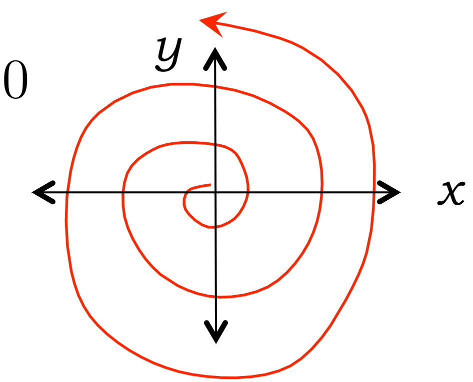
Stability?

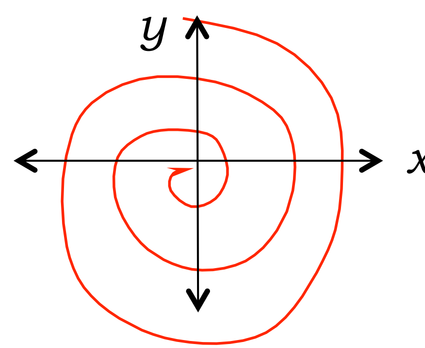
if $a > 0$ then $\mathrm{Re}(s) > 0$ and so $|\tilde{x}|,|\tilde{y}| \to \infty$ - linear instability

if $a < 0$ then $\mathrm{Re}(s) < 0$ and so $|\tilde{x}|,|\tilde{y}| \to 0$ - linear stability

But s is complex: the system oscillates toward zero or infinity.

$a > 0$





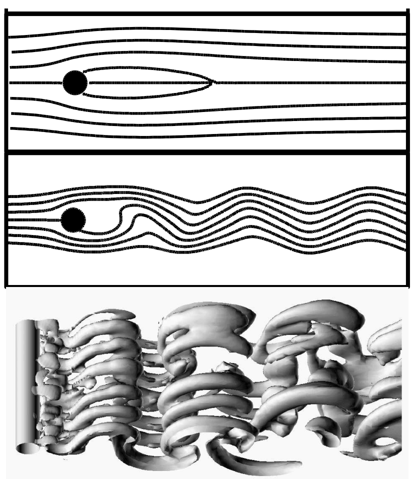
$a < 0$

But, the non-linear system has an unsteady, periodic, stable, base state:

$$\frac{dx}{dt} = -y + (a - x^2 - y^2)x \qquad x = \sqrt{a}\cos(t + t_0)$$

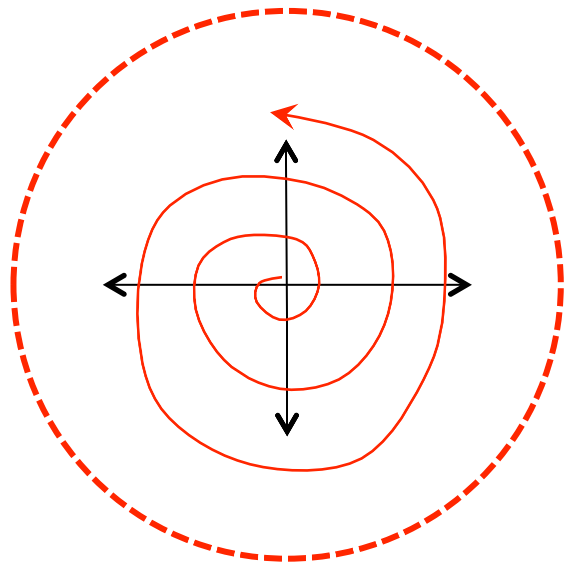
$$\frac{dy}{dt} = x + (a - x^2 - y^2)y \qquad y = \sqrt{a}\sin(t + t_0)$$

A higher-dimensional version of a supercritical pitchfork bifurcation,



1^{st} Supercritical Hopf bifurcation

2^{nd} Supercritical Hopf bifurcation



Bifurcation in the Lorenz system

The Lorenz equations

$$\frac{dx}{dt} = -\sigma(x-y)$$
$$\frac{dy}{dt} = rx - y - xz$$
$$\frac{dz}{dt} = -bz + xy$$

We will keep σ and b fixed and use r as the stability parameter.

The Lorenz equations

$$\frac{dx}{dt} = -\sigma(x - y)$$
$$\frac{dy}{dt} = rx - y - xz$$
$$\frac{dz}{dt} = -bz + xy$$

Steady state 1

$$(x_{B1}, y_{B1}, z_{B1}) = (0, 0, 0)$$

The Lorenz equations

$$\frac{dx}{dt} = -\sigma(x-y)$$
$$\frac{dy}{dt} = rx - y - xz$$
$$\frac{dz}{dt} = -bz + xy$$

*

Steady state 2

$$\frac{dx}{dt} = 0 \text{ gives } x = y$$
$$\frac{dy}{dt} = 0 \text{ gives } x(r-1) - xz = 0$$
$$\frac{dz}{dt} = 0 \text{ gives } -bz + x^2 = 0$$

Bifurcation in the Lorenz system

$$\frac{dx}{dt} = 0 \text{ gives } x = y$$

$$\frac{dy}{dt} = 0 \text{ gives } x(r-1) - xz = 0$$

$$\frac{dz}{dt} = 0 \text{ gives } -bz + x^2 = 0$$

$$z = r - 1.$$

$$x^2 = b(r-1)$$

*

Steady state 2

$$(x_{B2}, y_{B2}, z_{B2}) = (\pm\sqrt{b(r-1}, \pm\sqrt{(}b(r-1), r-1)$$

The Lorenz equations,

$$\frac{dx}{dt} = -\sigma(x-y)$$
$$\frac{dy}{dt} = rx - y - xz$$
$$\frac{dz}{dt} = -bz + xy$$

have steady states, also called **fixed points**:

$$(x_{B1}, y_{B1}, z_{B1}) = (0,0,0)$$

$$(x_{B2}, y_{B2}, z_{B2}) = (\pm\sqrt{b(r-1}, \pm\sqrt{(}b(r-1), r-1)$$

Base flow + perturbation

$$\frac{d\tilde{x}}{dt} = \sigma(\tilde{y} - \tilde{x})$$
$$\frac{d\tilde{y}}{dt} = r\tilde{x} - \tilde{y} - x_B\tilde{z} - z_B\tilde{x}$$
$$\frac{d\tilde{z}}{dt} = -b\tilde{z} + x_B\tilde{y} + y_B\tilde{x}$$

Linear stability for $(x_{B1}, y_{B1}, z_{B1}) = (0, 0, 0)$

$$\frac{d\tilde{x}}{dt} = \sigma(\tilde{y} - \tilde{x})$$
$$\frac{d\tilde{y}}{dt} = r\tilde{x} - \tilde{y}$$
$$\frac{d\tilde{z}}{dt} = -b\tilde{z}$$

The linearised Lorenz equations,

$$\frac{d\tilde{x}}{dt} = \sigma(\tilde{y} - \tilde{x})$$
$$\frac{d\tilde{y}}{dt} = r\tilde{x} - \tilde{y}$$
$$\frac{d\tilde{z}}{dt} = -b\tilde{z}$$

The third equation is uncoupled from the first two and produces exponential decay (first eigenvalue =-b),

$$\tilde{z} = \gamma \mathrm{e}^{-bt}$$

The first two equations are coupled and so solutions must be sought for

$$\tilde{x} = \alpha \mathrm{e}^{st}$$
$$\tilde{y} = \beta \mathrm{e}^{st}$$

Inserting in linearised Lorenz equations,

$$\tilde{x} = \alpha e^{st} \qquad \tilde{y} = \beta e^{st} \longrightarrow \frac{d\tilde{x}}{dt} = \sigma(\tilde{y} - \tilde{z}), \quad \frac{d\tilde{y}}{dt} = r\tilde{x} - \tilde{y}, \quad \frac{d\tilde{z}}{dt} = -b\tilde{z}$$

Leads to the **eigenvalue problem**

$$\alpha s = \sigma(\beta - \alpha),$$
$$\beta s = r\alpha - \beta.$$

This has non-trivial solution when det[A-sI]=0

$$\longrightarrow (s + \sigma)(s + 1) - \sigma r = 0$$

Second and third eigenvalues given by:

$$\tilde{x} = \alpha \mathrm{e}^{st}$$
$$\tilde{y} = \beta \mathrm{e}^{st}$$

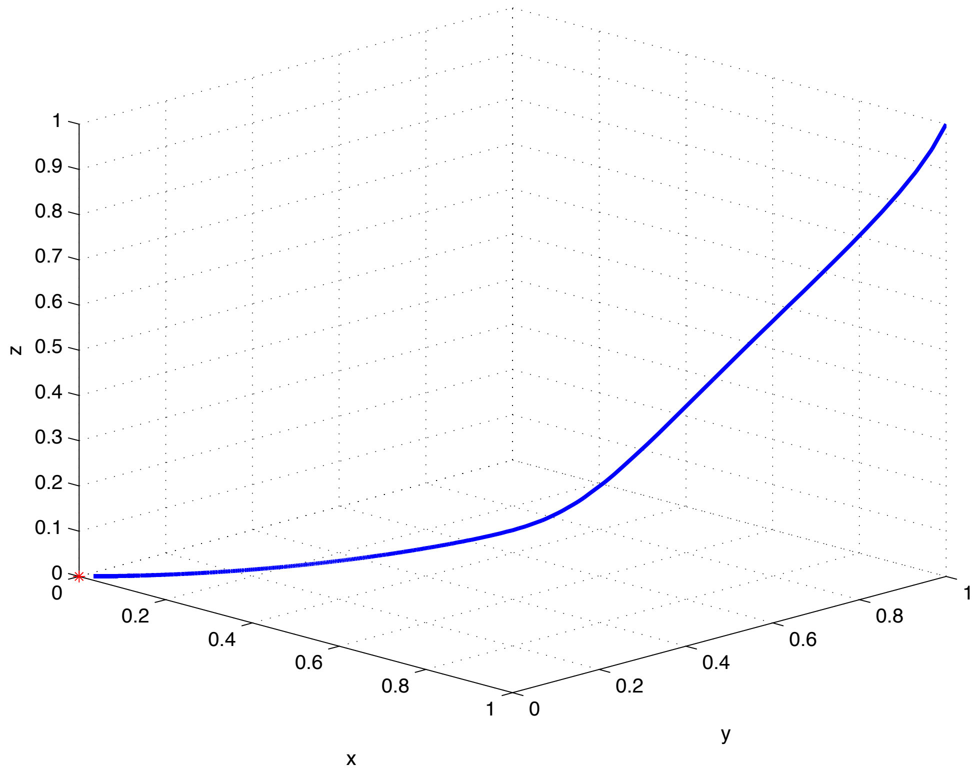
$$(s+\sigma)(s+1) - \sigma r = 0$$

$$s = \frac{1}{2}\left[-(\sigma+1) \pm \sqrt{(\sigma+1)^2 - 4\sigma(1-r)}\right]$$

*

Stable for *r<1*, unstable for *r>1*

r=0.6



$$\tilde{x} = \alpha \mathrm{e}^{st}$$
$$\tilde{y} = \beta \mathrm{e}^{st}$$

$$(s+\sigma)(s+1) - \sigma r = 0$$

$$s = \frac{1}{2}\Big[-(\sigma+1) \pm \sqrt{(\sigma+1)^2 - 4\sigma(1-r)}\Big]$$

Stable for *r<1*, unstable for *r>1*,

But we know that a second base state comes into existance at *r>1*:

$$(x_{B2}, y_{B2}, z_{B2}) = (\pm\sqrt{b(r-1}, \pm\sqrt{(}b(r-1), r-1)$$

So *r=1* is supercritical pitchfork bifurcation

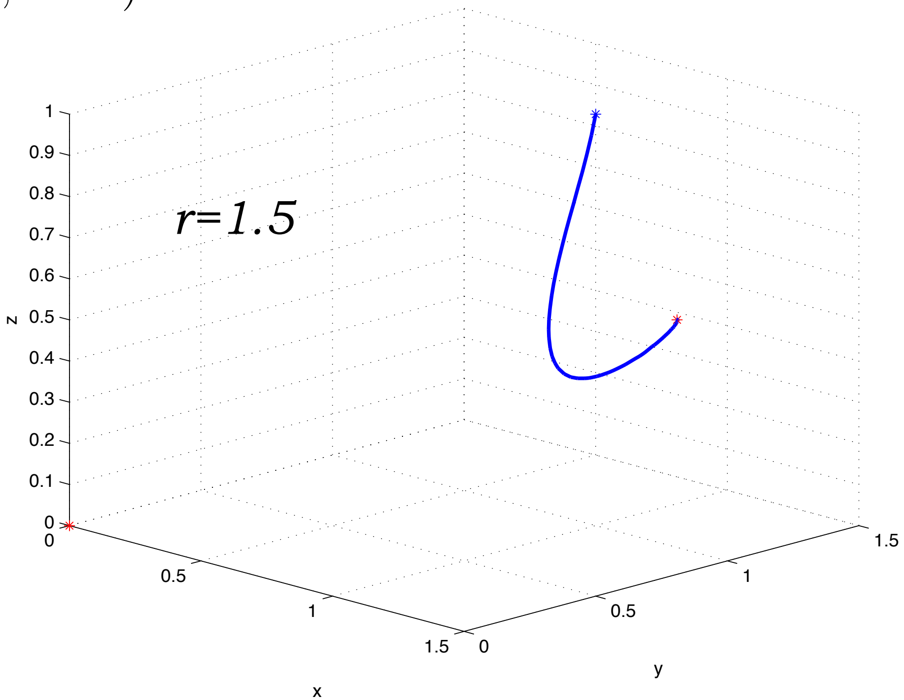
Bifurcation in the Lorenz system

$$\tilde{x} = \alpha \mathrm{e}^{st}$$
$$\tilde{y} = \beta \mathrm{e}^{st}$$

$$(s+\sigma)(s+1) - \sigma r = 0$$

$$s = \frac{1}{2}\Big[-(\sigma+1) \pm \sqrt{(\sigma+1)^2 - 4\sigma(1-r)} \Big]$$

$$(x_{B2}, y_{B2}, z_{B2}) = (\pm\sqrt{b(r-1}, \pm\sqrt{(b(r-1)}, r-1)$$



What are the stability characteristics of:

$$(x_{B2}, y_{B2}, z_{B2}) = (\pm\sqrt{b(r-1}, \pm\sqrt{(}b(r-1), r-1)$$

Inserting base state + perturbation into the governing equation, linearising and searching for solutions of the form:

$$\tilde{x} = \alpha \mathrm{e}^{st}$$
$$\tilde{y} = \beta \mathrm{e}^{st}$$
$$\tilde{z} = \gamma \mathrm{e}^{st}$$

leads to

$$s^3 + (\sigma + b + 1)s^2 + b(\sigma + r)s + 2b\sigma(r-1) = 0$$

$$s^3 + (\sigma + b + 1)s^2 + b(\sigma + r)s + 2b\sigma(r - 1) = 0$$

Cubic equation: two possiblities:

- 1 real eigenvalue + 1 complex conjugate pair,
- 3 real eigenvalues.

If 3 real eigenvalues: exponential growth or decay, from or toward the fixed point,

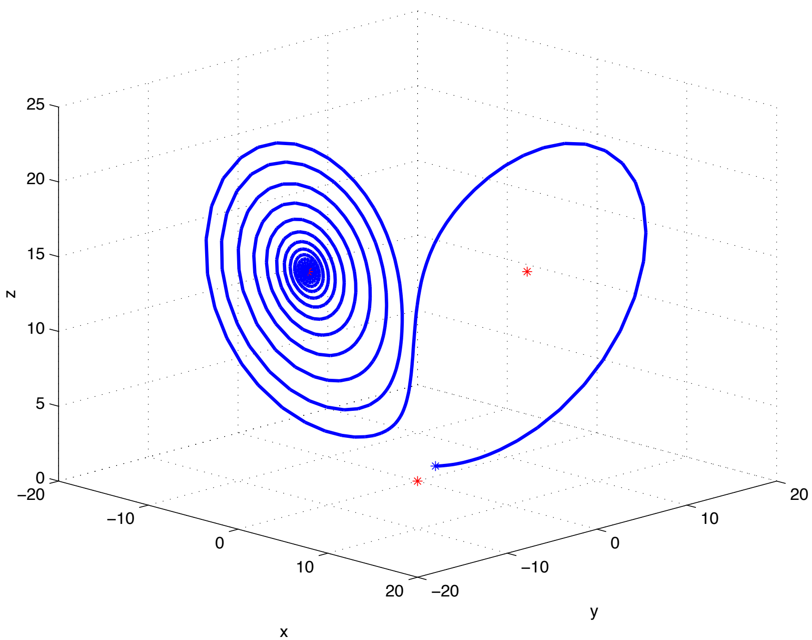
If 1 real eigenvalue + 1 complex conjugate pair:

- 1 real eigenplane (from the real and imaginary parts of the complex eigenvector),
- 1 real eigendirection (sketch too difficult for powerpoint ! ...do it on the blackboard.)

$$s^3 + (\sigma + b + 1)s^2 + b(\sigma + r)s + 2b\sigma(r - 1) = 0$$

If 1 real **NEGATIVE** eigenvalue + 1 complex conjugate pair with **NEGATIVE** real part:

- 1 **real stable eigenplane** (from the real and imaginary parts of the complex eigenvector),
- 1 real eigendirection.

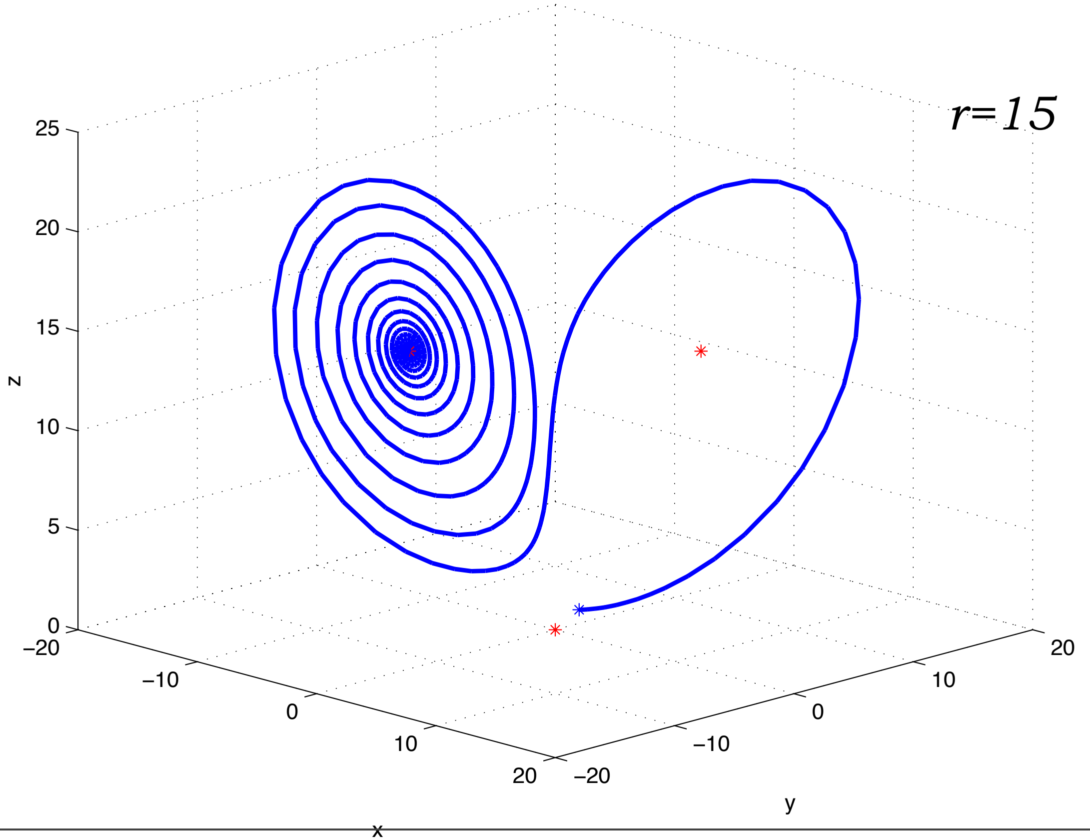


Basins of attraction:
- one set of ICs will go to one attractor, another set will go to the other attractor.

Bifurcation in the Lorenz system

$$s^3 + (\sigma + b + 1)s^2 + b(\sigma + r)s + 2b\sigma(r - 1) = 0$$

At *r=24,74* the real part of the complex eigenvalues changes sign, and the system becomes unstable in the eigenplane; the real eigenvalue remains negative...

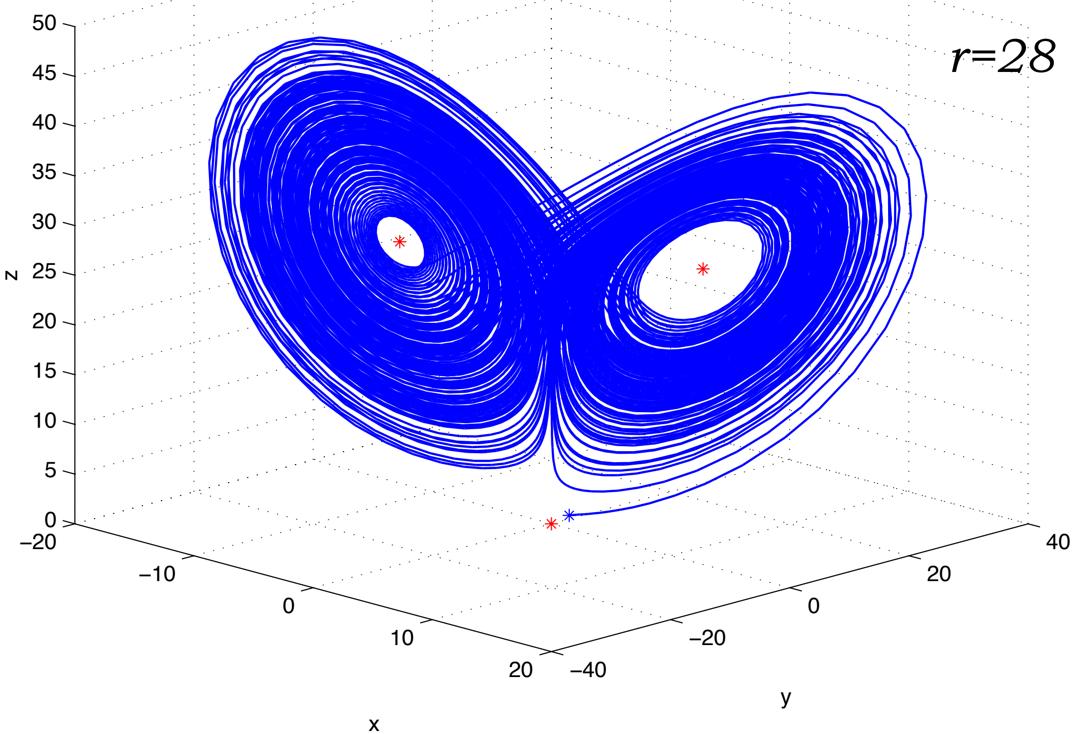


Bifurcation in the Lorenz system

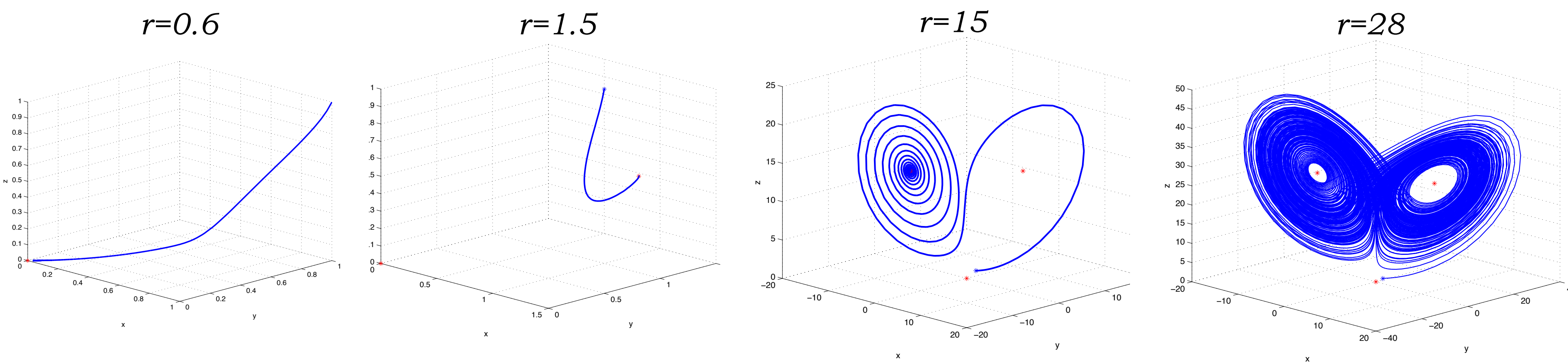
$$r_c = \frac{\sigma(3+b+\sigma)}{\sigma - b - 1}$$

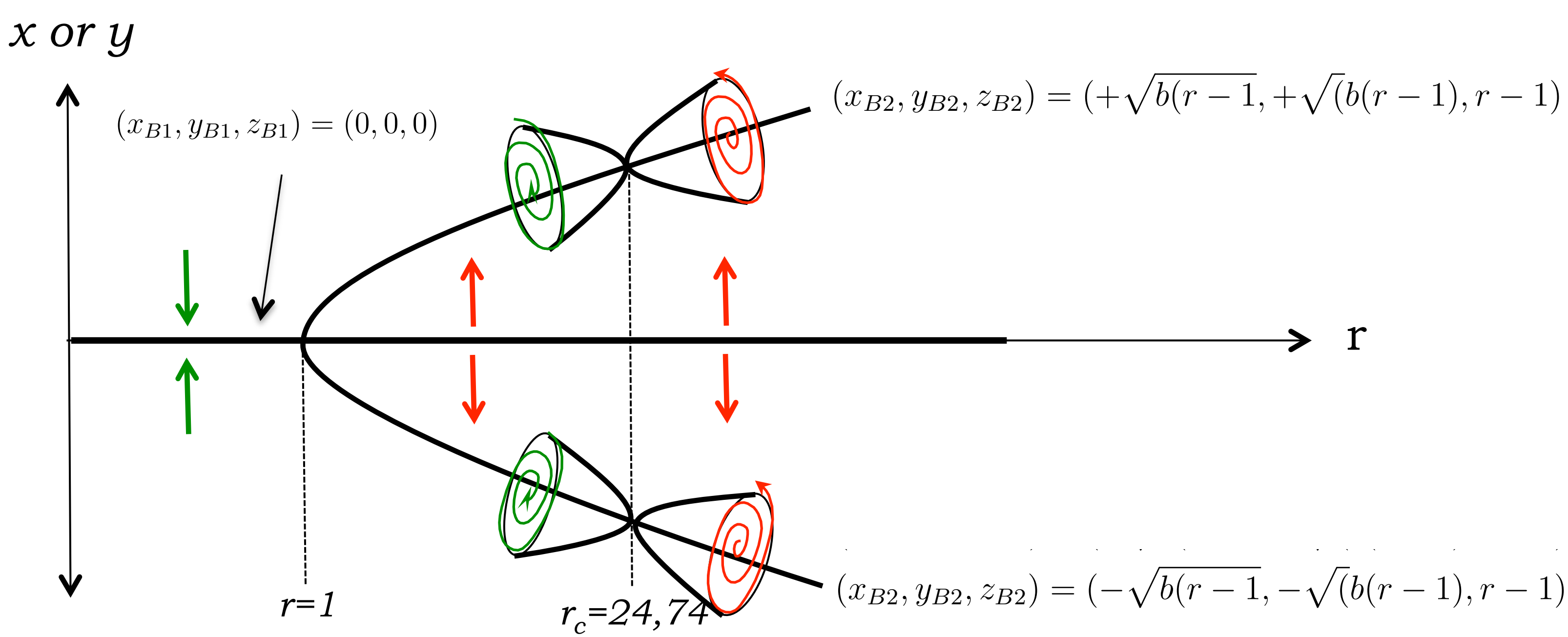
=24,74 for the parameters used in the model.

$$(x_{B2}, y_{B2}, z_{B2}) = (\pm\sqrt{b(r-1}, \pm\sqrt{(}b(r-1), r-1)$$



Bifurcation in the Lorenz system





Résumé

Linear analysis allows the identification of:

- neutral curves,
- bifurcation points,
- bifurcation characteristics.

Identification of the various base states permitted by the non-linear equation, followed by linear stability analysis of these, gives us a means by which to construct a skeleton of the stability characteristics through a variety of bifurcations.

This skeleton is a departure point for constructing the manifold on which the non-linear dynamics evolves.