I N S T I T U T P P R I M E
CNRS–UPR–3346 • UNIVERSITÉ DE POITIERS • ENSMA

DÉPARTEMENT D2 – FLUIDES
THERMIQUE ET COMBUSTION

An introduction to hydrodynamic stability

Lecture 6: Non-modal stability

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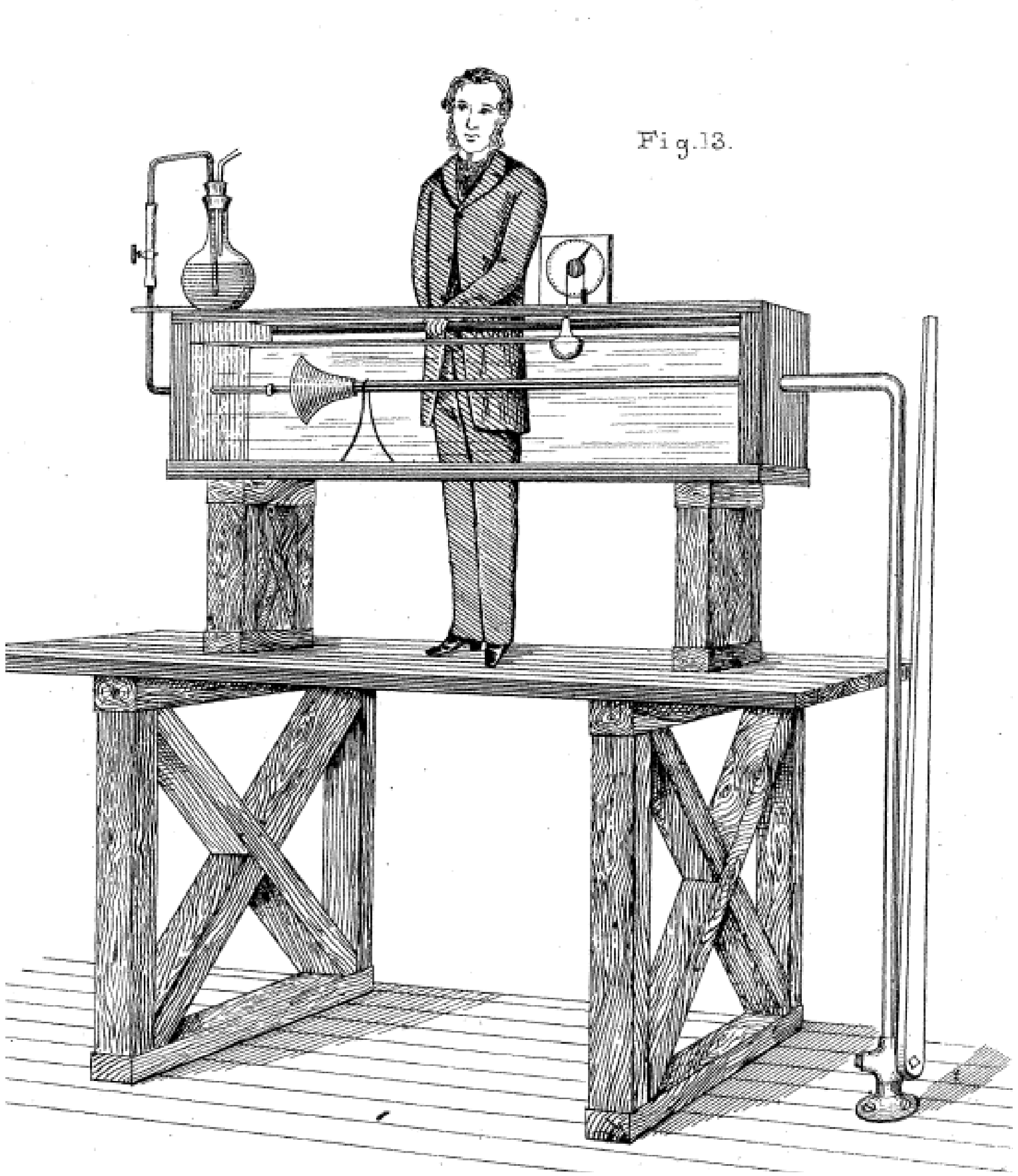
1. The enigma of pipe flow
2. The Orr-Sommerfeld-Squire system
3. The initial-value problem
4. Non-normality and transient growth
5. ~~Bi-orthogonal projection~~
6. Optimal transient growth
7. Resolvent analysis

1. The enigma of pipe flow



Linear stability analysis of pipe flow,

$$U(r) = (1 - r^2)$$

shows that:

- it is stable to inviscid disturbances,
- it is stable to viscous disturbances,

But this is precisely *the* first experiment to demonstrate laminar-turbulent transition at

$Re \approx 2300$!?

1. The enigma of pipe flow

Eckhardt et al. ARFM 2007

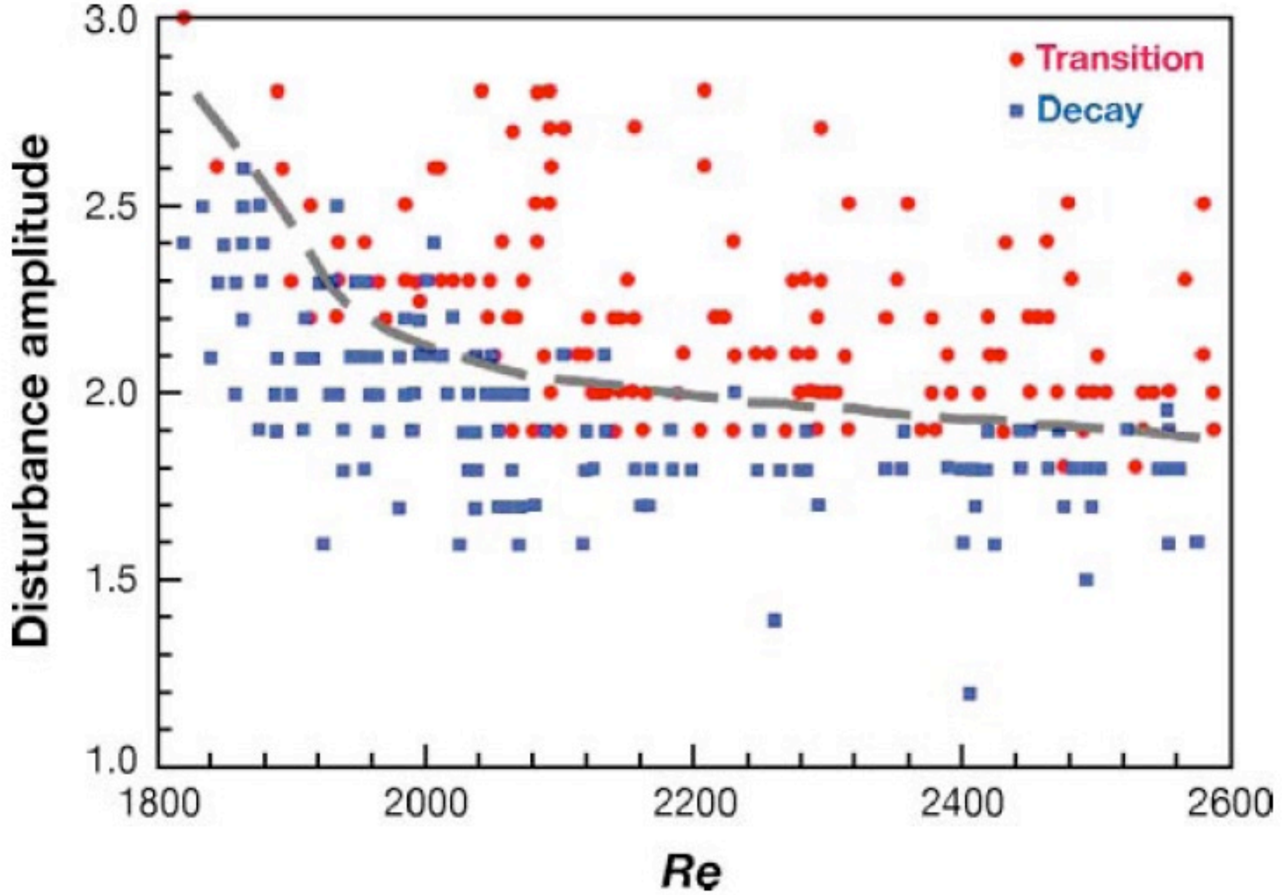


Figure 1

Transition experiments by Darbyshire & Mullin (1995). Disturbances were introduced at a distance 70 diameters downstream of the inlet, and their status was probed at another 120 diameters downstream, delayed with the mean advection time. Depending on whether the perturbation was still present or not, a point was marked “transition” or “decay.” The amplitude of the perturbations is proportional to the injected fluid volume. For more details, see Darbyshire & Mullin (1995). Redrawn after Darbyshire & Mullin (1995).

Hint: transition in pipe flow depends on the amplitude of disturbances

Research continues on the Reynolds experiment

1. The enigma of pipe flow

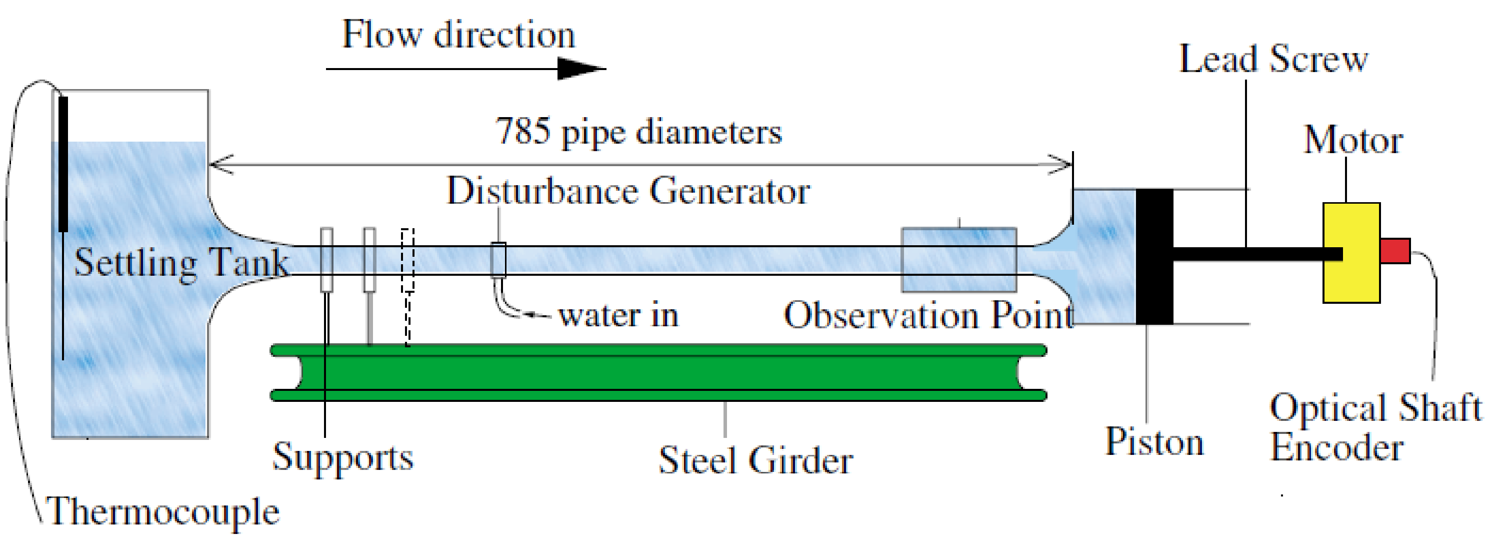


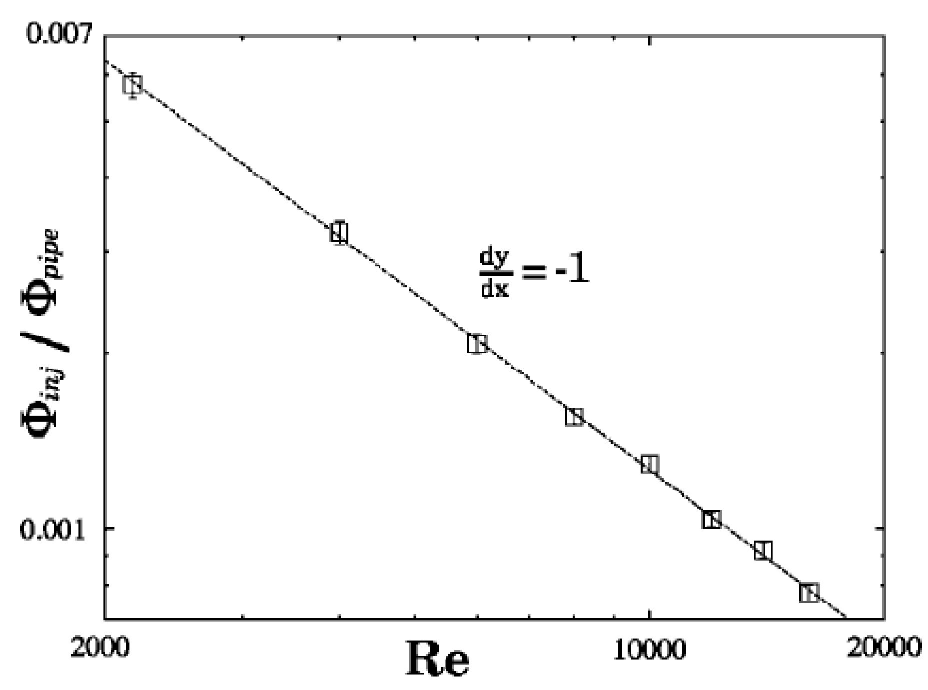
FIG. 1 (color online). Schematic of the long pipe experimental system.

Hint: transition in pipe flow depends on the amplitude of disturbances

Hof et al. Phy. Rev. 2003

Pipe is 15m long!

Normalised disturbance amplitude



We've seen something like this before

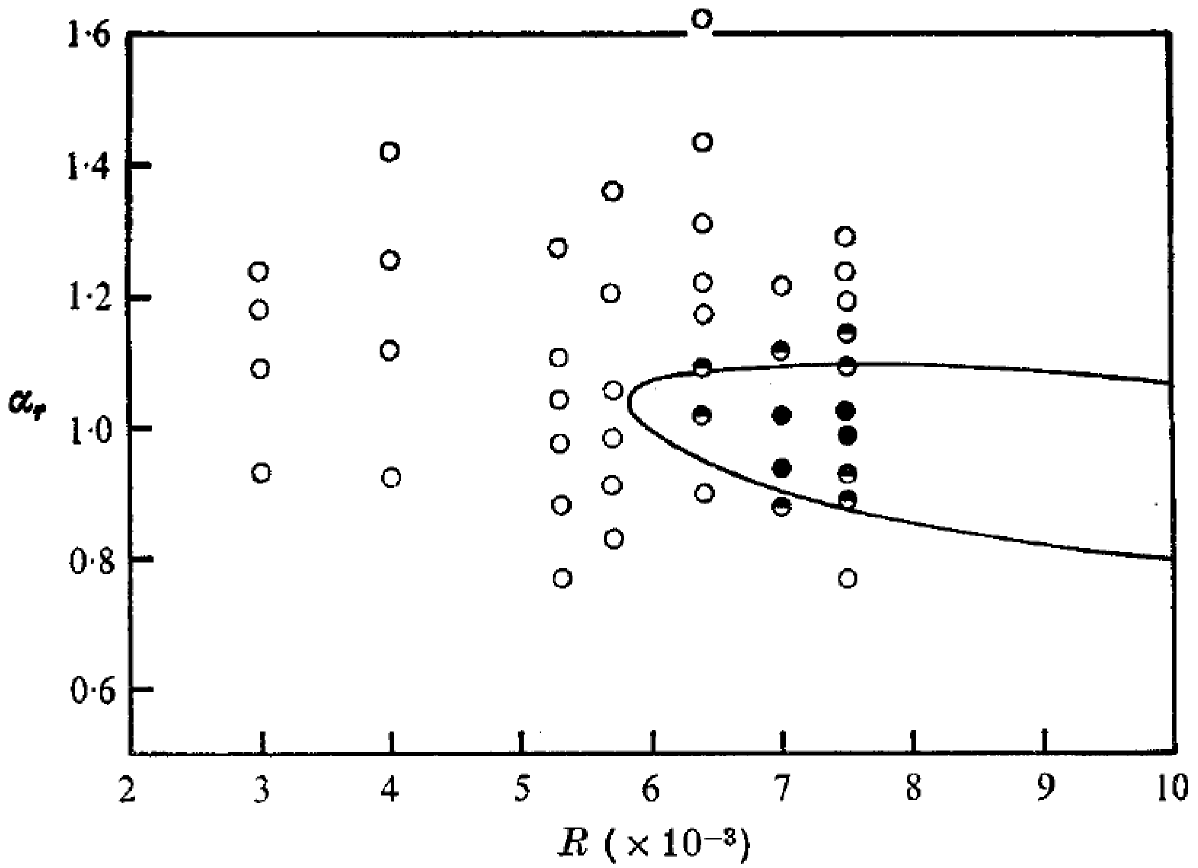


FIGURE 11. Stability boundary for small disturbances. ——, Ito's neutral curve. Our results: ○, damped; ◓, nearly neutral; ●, amplified.

Patel & Head 1969: $Re_c = 2500$

Karnitz et al. 1974 $Re_c = 5000$
(background disturbances: 0.3%)

Classical modal linear stability analysis only worked for Nishioka et al. because of very low background levels

So far we've been focused on a mode-by-mode study (modal approach)

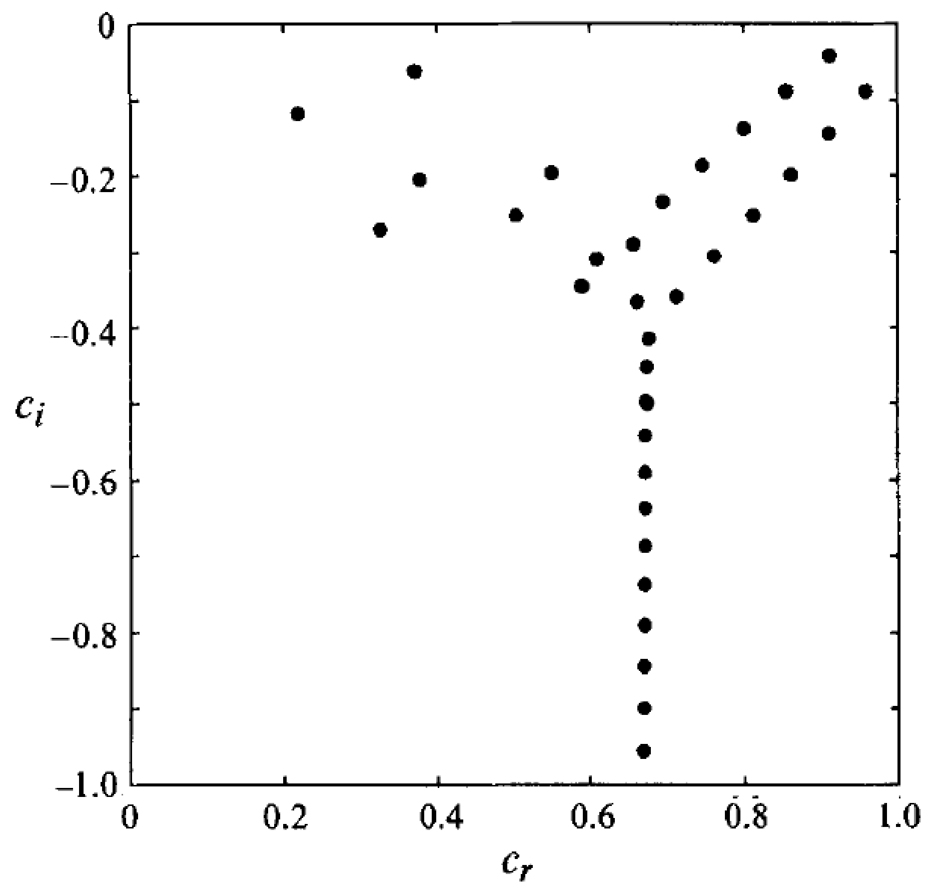


FIGURE 1. Pipe flow spectrum Λ_R for $\alpha = 1$, $n = 1$, $R = 3000$. $N = 60$ Chebyshev polynomials have been used to discretize the normal direction.

Mode unstable if $c_I > 0$

What about linear combinations of modes?

Will their linear combination grow or decay in time?

Ellingsen & Palm 1975

Linearised Navier-Stokes for uniform shear flow with no streamwise gradients

$$\frac{\partial u}{\partial t} + \frac{\partial U}{\partial y} v = 0,$$

$$\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y},$$

$$\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z},$$

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Define stream function, ψ

$$v = \frac{\partial \psi}{\partial z} \quad ; \quad w = -\frac{\partial \psi}{\partial y}$$

Linearised Navier-Stokes for uniform shear flow with no streamwise gradients

$$v = \frac{\partial \psi}{\partial z} \quad ; \quad w = -\frac{\partial \psi}{\partial y} \quad \longrightarrow \quad \begin{aligned} \frac{\partial^2 \psi}{\partial t \partial z} &= -\frac{\partial p}{\partial y}, \\ \frac{\partial^2 \psi}{\partial t \partial y} &= \frac{\partial p}{\partial z} \end{aligned}$$

$$\frac{\partial}{\partial z}\left[\frac{\partial^2 \psi}{\partial t \partial z} = -\frac{\partial p}{\partial y}\right] + \frac{\partial}{\partial y}\left[\frac{\partial^2 \psi}{\partial t \partial y} = \frac{\partial p}{\partial z}\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\nabla^2 \psi = 0$$

implying that ψ is constant in time, as are its components, v & w

Linearised Navier-Stokes for uniform shear flow with no streamwise gradients

implying that ψ is constant in time, as are its components, v & w

$$\frac{\partial u}{\partial t} + \frac{\partial U}{\partial y} v = 0,$$

Constant !

$$\longrightarrow \quad u(t) = u(t=0) - \frac{\partial U}{\partial y} v t$$

Which is an algebraic instability in time!

2. The Orr-Sommerfeld-Squire system

Besides Orr-Sommerfeld

$$\left[(-i\omega + i\alpha U)(D^2 - \alpha^2 - \beta^2) - i\alpha \frac{\mathrm{d}^2 U}{\mathrm{d}y^2} - \frac{1}{Re}(D^2 - \alpha^2 - \beta^2)^2\right]\hat{v} = 0$$

from linearised Navier-Stokes we can derive an equation for the normal vorticity,

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

called the Squire equation

$$\left[(-i\omega + i\alpha U) - \frac{1}{Re}(D^2 - \alpha^2 - \beta^2)\right]\eta = -i\beta \frac{\mathrm{d}U}{\mathrm{d}y}\hat{v}$$

2. The Orr-Sommerfeld-Squire system

O-S
$$\left[(-i\omega + i\alpha U)(D^2 - \alpha^2 - \beta^2) - i\alpha \frac{\mathrm{d}^2 U}{\mathrm{d}y^2} - \frac{1}{Re}(D^2 - \alpha^2 - \beta^2)^2\right]\hat{v} = 0$$

Squire
$$\left[(-i\omega + i\alpha U) - \frac{1}{Re}(D^2 - \alpha^2 - \beta^2)\right]\eta = -i\beta \frac{\mathrm{d}U}{\mathrm{d}y}\hat{v}$$

$\eta = 0$ **on a wall**

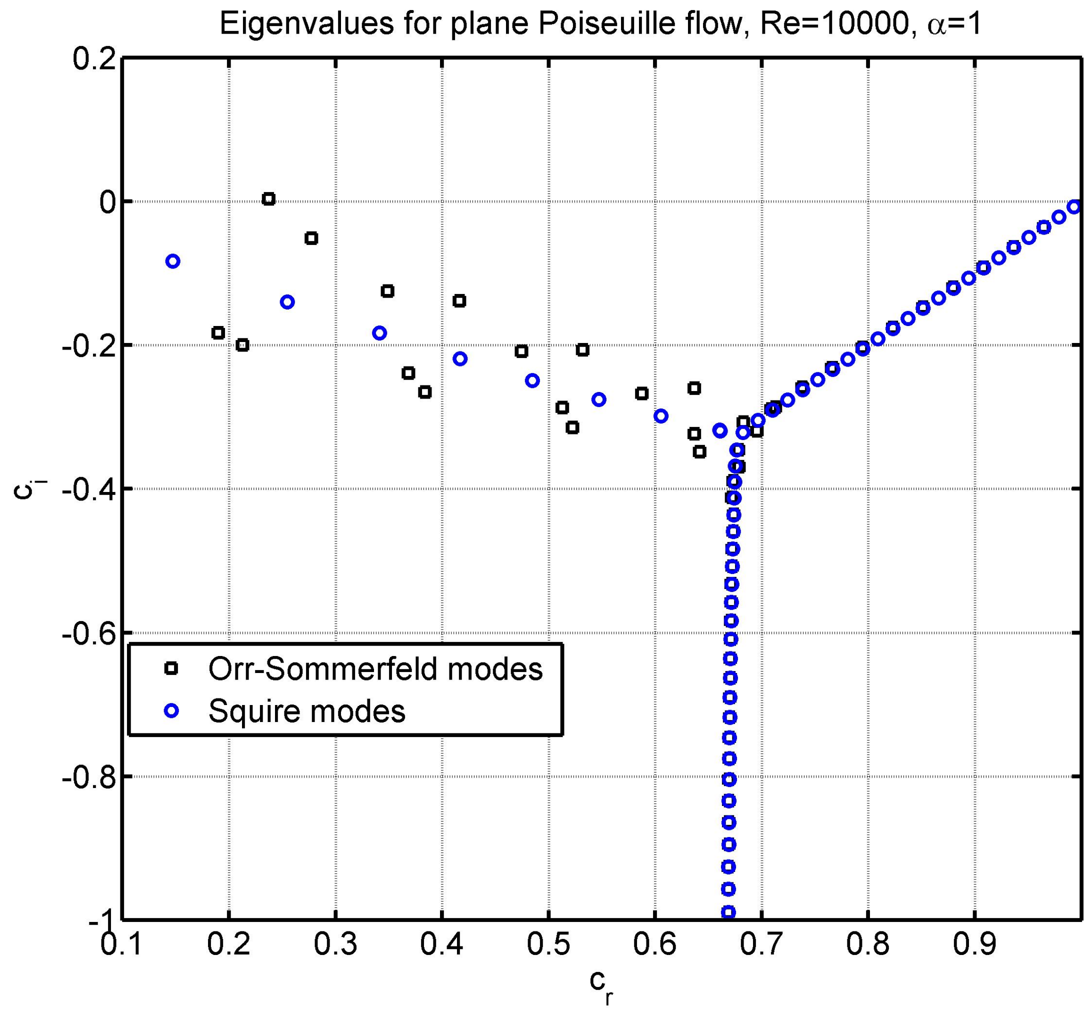
Note that O-S is decoupled from Squire and can be solved separately

Two sets of modes are obtained:

- **O-S modes: solve eigenvalue problem for** v
- **Squire modes: let ,** $v = 0$ **and compute modes for** η

Exercise: obtain Squire eigenvalues for plane Poiseuille flow. Can you find an unstable Squire mode?

2. The Orr-Sommerfeld-Squire system



All Squire modes are stable

This is why O-S alone is sufficient to assess the modal stability of the system

So, why do we care about the Squire equation?

2. The Orr-Sommerfeld-Squire system

O-S
$$\left[(-i\omega + i\alpha U)(D^2 - \alpha^2 - \beta^2) - i\alpha \frac{\mathrm{d}^2 U}{\mathrm{d}y^2} - \frac{1}{Re}(D^2 - \alpha^2 - \beta^2)^2\right]\hat{v} = 0$$

Squire
$$\left[(-i\omega + i\alpha U) - \frac{1}{Re}(D^2 - \alpha^2 - \beta^2)\right]\eta = -i\beta \frac{\mathrm{d}U}{\mathrm{d}y}\hat{v}$$

In the time domain these can be written as,

$$\frac{d}{dt}\begin{pmatrix}\mathbf{v} \\ \eta\end{pmatrix} = \underbrace{\begin{pmatrix}\mathsf{L}_{\mathsf{OS}} & 0 \\ \mathsf{L}_{\mathsf{C}} & \mathsf{L}_{\mathsf{SQ}}\end{pmatrix}}_{\mathsf{L}}\begin{pmatrix}\mathbf{v} \\ \eta\end{pmatrix}$$

Schmidt & Brandt 2014

$$\mathsf{L}_{\mathsf{OS}} = \mathsf{M}^{-1}\left(-i\alpha U\mathsf{M} - i\alpha U'' - \frac{1}{Re}\mathsf{M}^2\right),$$
$$\mathsf{L}_{\mathsf{SQ}} = -i\alpha U - \frac{1}{Re}\mathsf{M},$$
$$\mathsf{L}_{\mathsf{C}} = -i\beta U',$$

$$M = k^2 - D^2$$

3. The initial-value problem

In the time domain these can be written as,

$$\frac{d}{dt}\begin{pmatrix}\mathbf{v}\\ \eta\end{pmatrix} = \underbrace{\begin{pmatrix}\mathrm{L_{OS}} & 0\\ \mathrm{L_C} & \mathrm{L_{SQ}}\end{pmatrix}}_{\mathrm{L}}\begin{pmatrix}\mathbf{v}\\ \eta\end{pmatrix}$$

To understand the key behaviour, consider a simpler system with two degrees of freedom

$$\frac{d}{dt}\begin{pmatrix}q_1\\ q_2\end{pmatrix} = \underbrace{\begin{pmatrix}\frac{1}{100}-\frac{1}{Re} & 0\\ \mu & -\frac{2}{Re}\end{pmatrix}}_{\mathrm{A}}\begin{pmatrix}q_1\\ q_2\end{pmatrix}$$

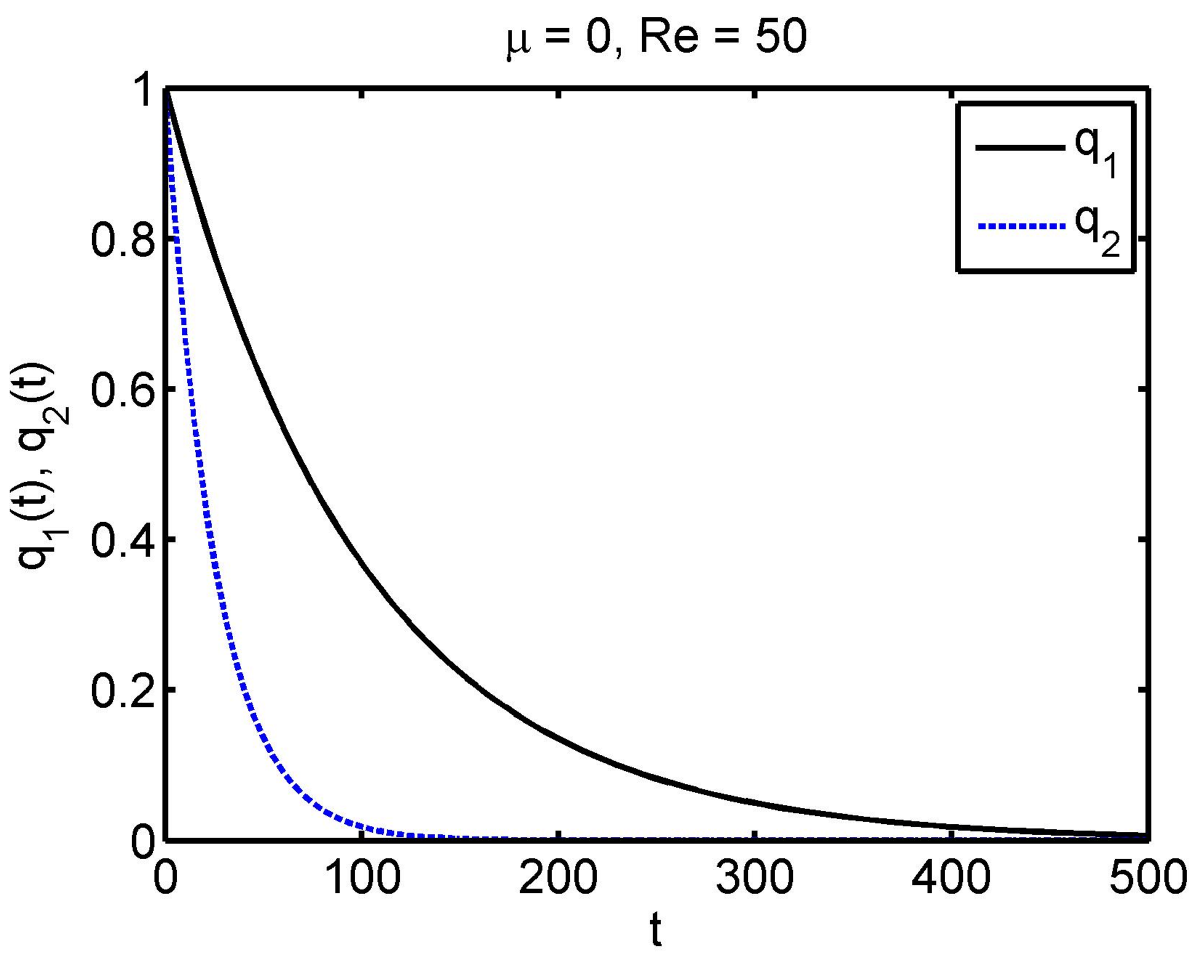
Exercise: determine the stability characteristics of the system using normal modes

Exercise: simulate the initial-value problem with different initial conditions and Reynolds numbers, for

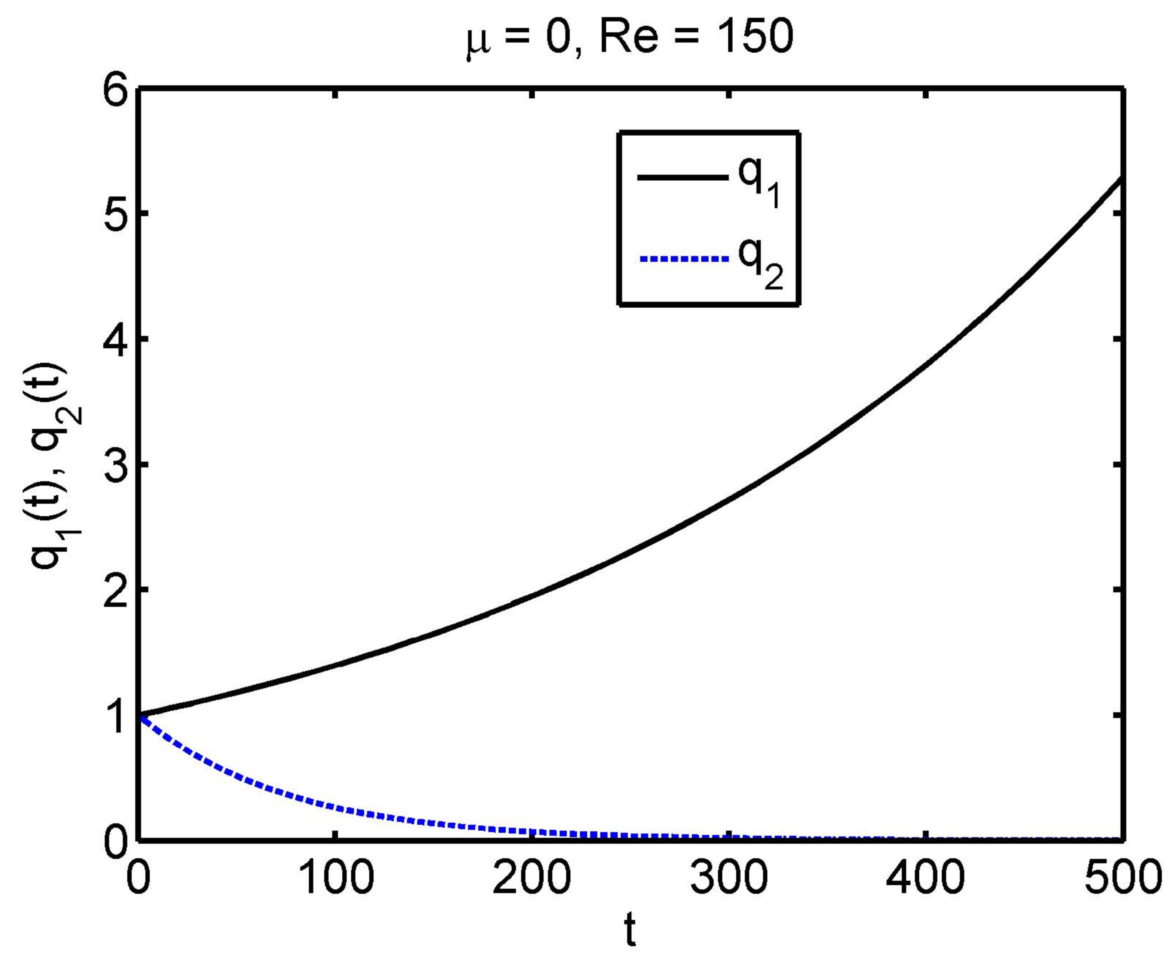
$$\mu = 0$$

$$\mu = 1$$

The normal system, $\mu = 0$

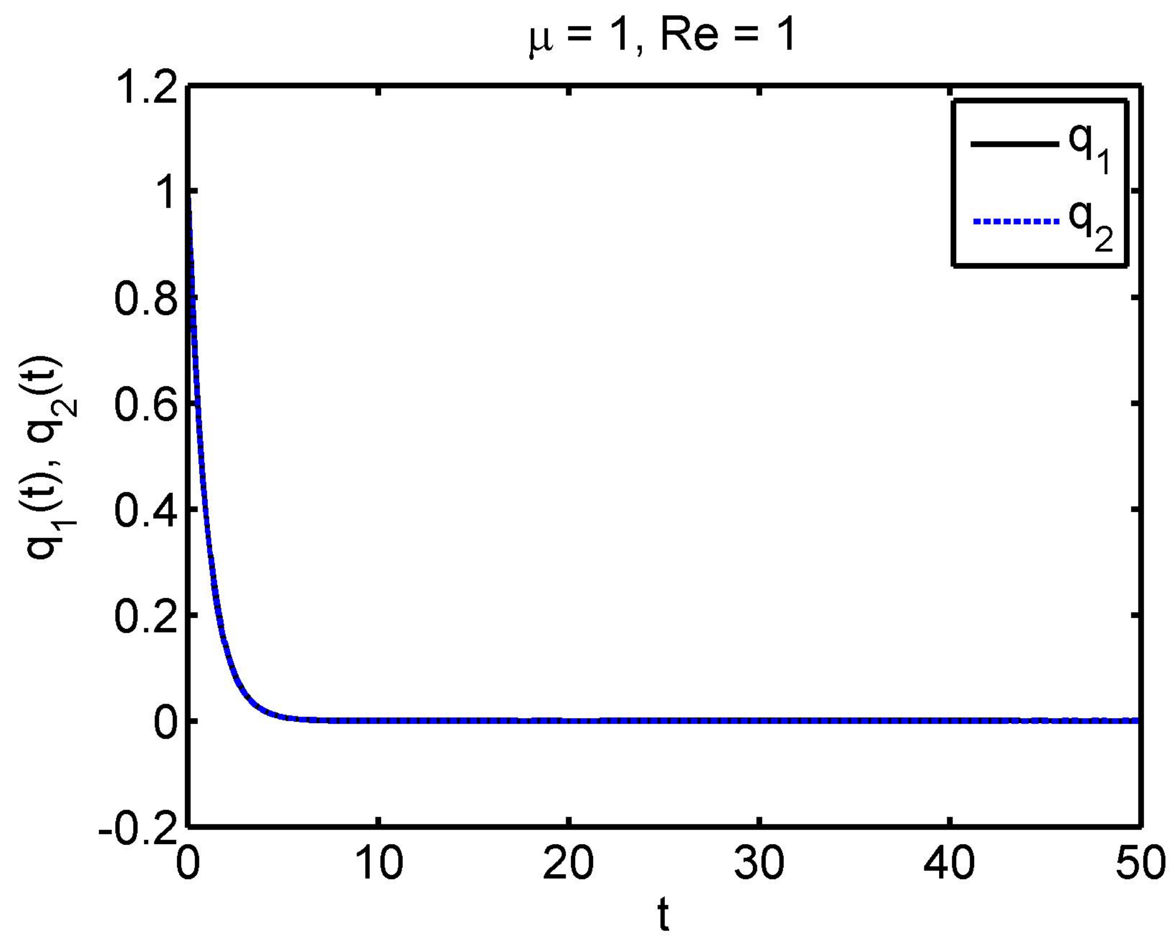


Stable



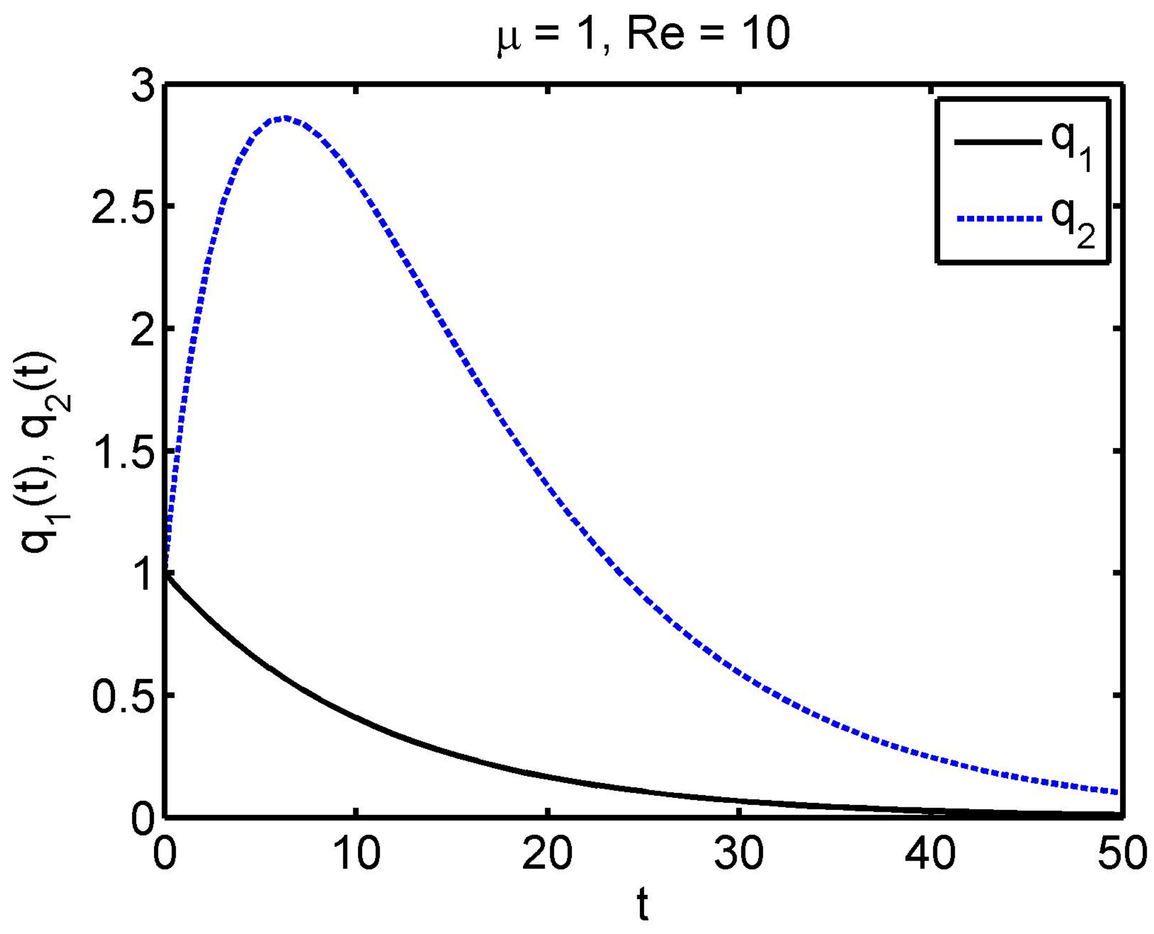
Unstable
- exponential (modal) growth

The non-normal system, $\mu = 1$



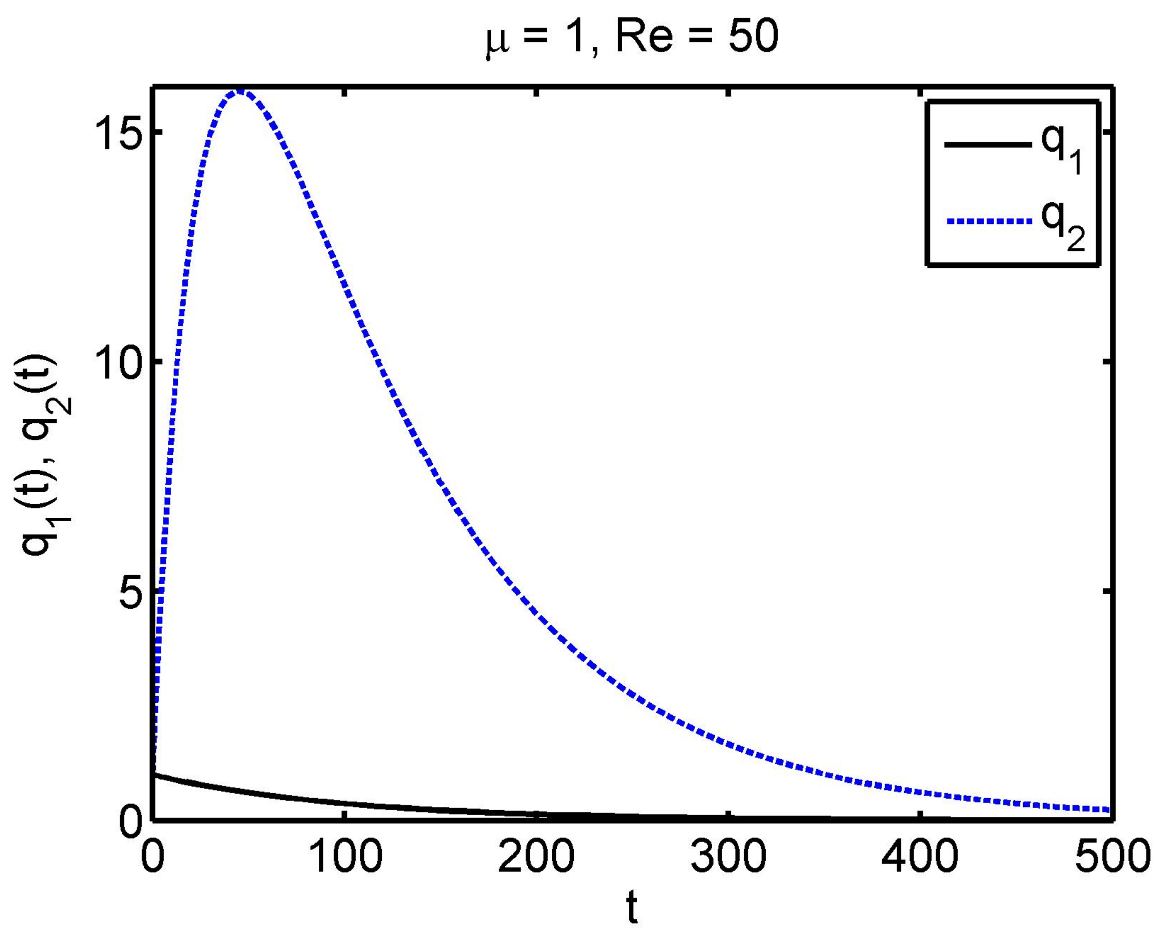
**Stable: no exponential growth,
perturbations tend asymptotically to zero.**

The non-normal system, $\mu = 1$



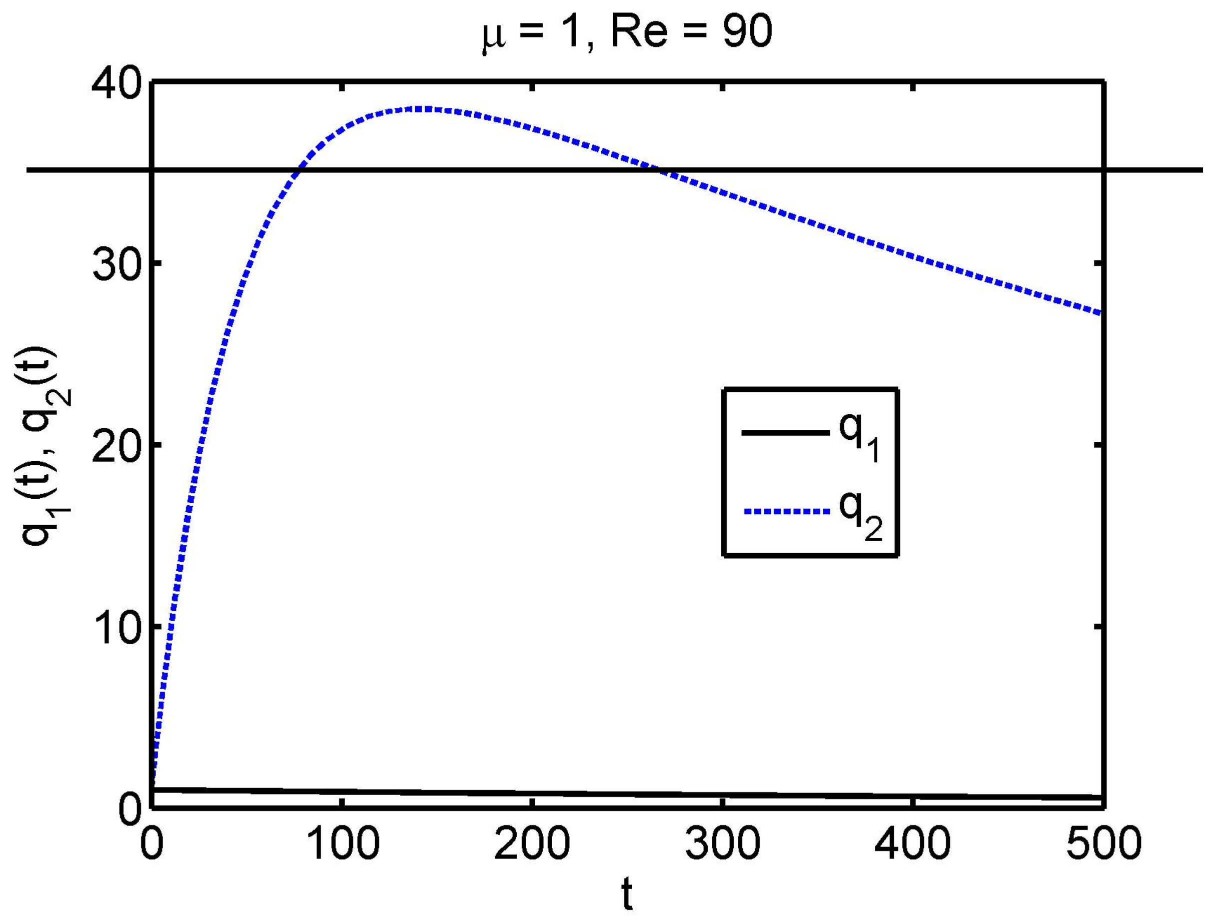
Stable: no exponential growth,
perturbations tend asymptotically to zero,
but an initial transient comprises growth

The non-normal system, $\mu = 1$



Stable: no exponential growth,
perturbations tend asymptotically to zero,
but an initial transient comprises *even larger* growth

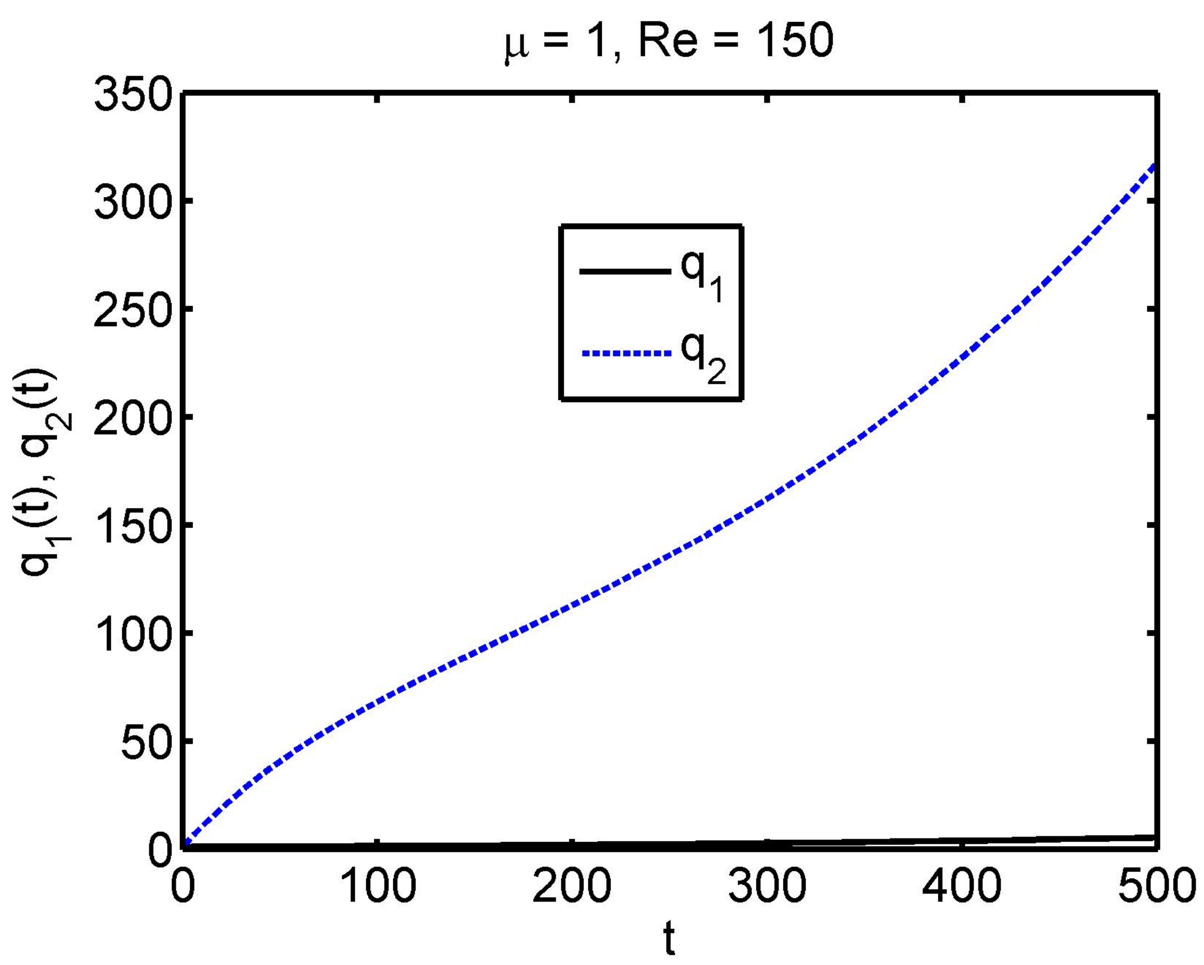
The non-normal system, $\mu = 1$



If threshold for 'activation' of non-linear terms lies here then transition will occur, despite the asymptotic linear stability of the system

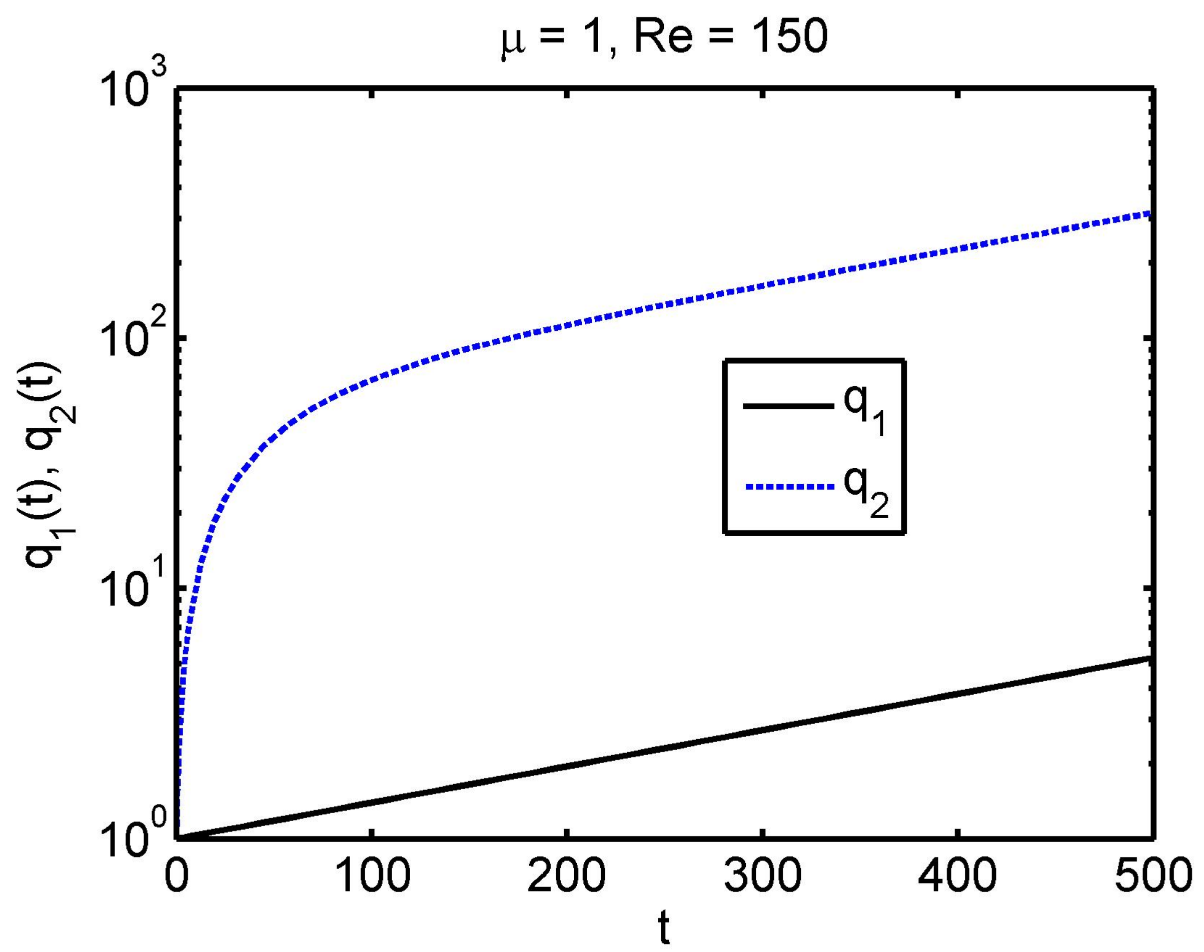
Stable: no exponential growth, perturbations tend asymptotically to zero, but an initial transient comprises *even larger* growth

The non-normal system, $\mu = 1$



Unstable: exponential growth,
but only after initial algebraic transient

The non-normal system, $\mu = 1$



Unstable: exponential growth,
but only after initial algebraic transient

4. Non-normality and transient growth

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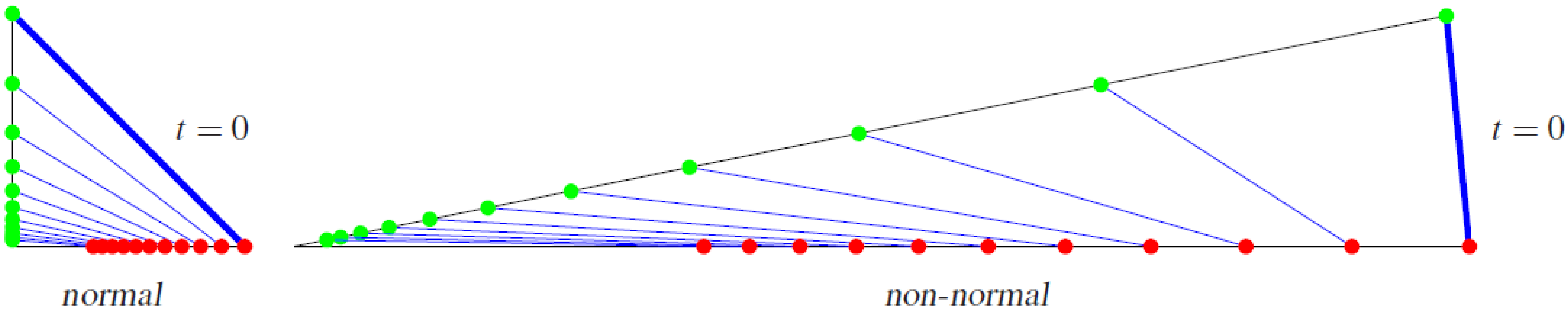


Fig. 3. Geometric interpretation of transient growth.

Normal systems (orthogonal modes): stability depends only on eigenvalues

Non-normal systems can experience transient growth due to non-orthogonality of eigenfunctions

Transient growth may transition to turbulence if disturbance amplitude is sufficiently high

4. Non-normality and transient growth

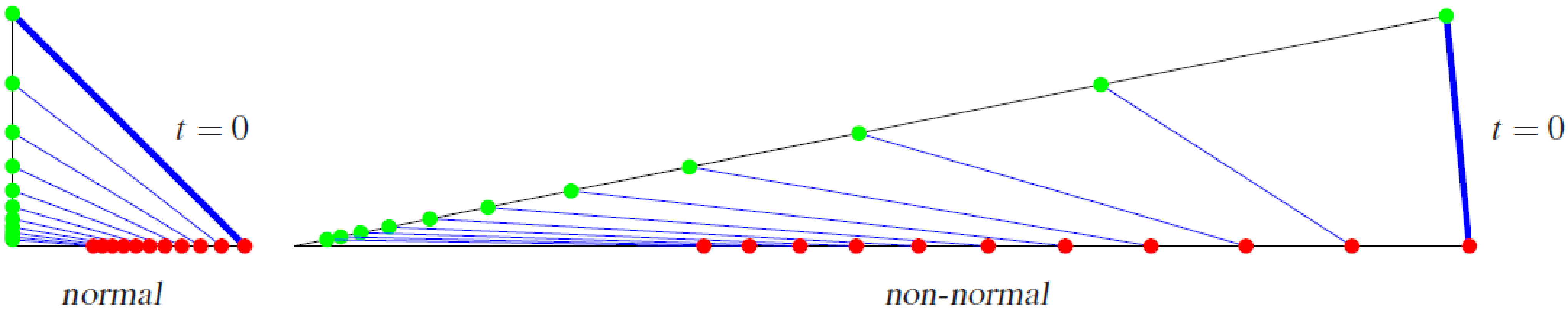


Fig. 3. Geometric interpretation of transient growth.

$$\frac{d}{dt}\begin{pmatrix} \mathbf{v} \\ \eta \end{pmatrix} = \underbrace{\begin{pmatrix} \mathsf{L_{OS}} & 0 \\ \mathsf{L_C} & \mathsf{L_{SQ}} \end{pmatrix}}_{\mathsf{L}} \begin{pmatrix} \mathbf{v} \\ \eta \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{100} - \frac{1}{Re} & 0 \\ \mu & -\frac{2}{Re} \end{pmatrix}}_{\mathsf{A}} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

4. Non-normality and transient growth

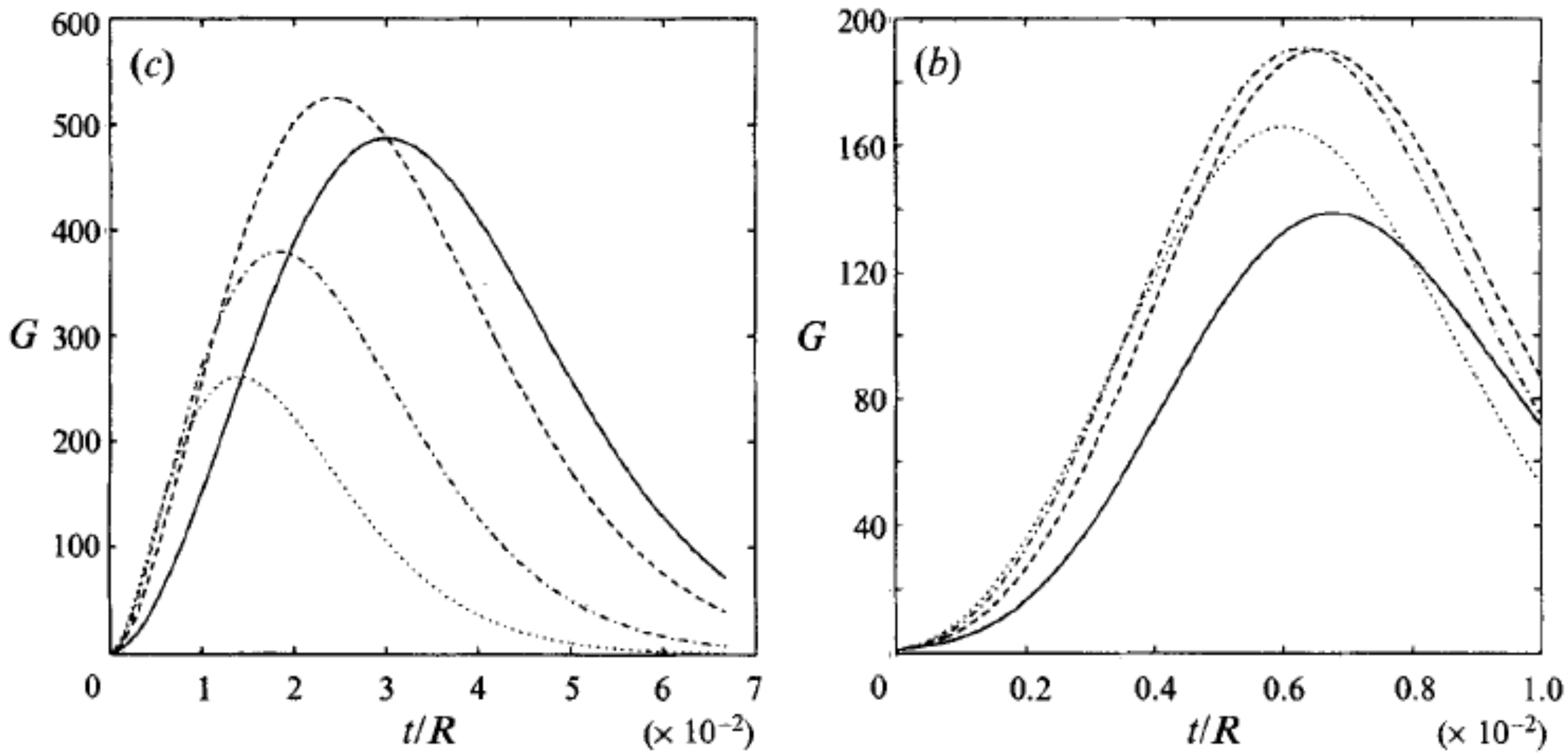


FIGURE 4. Transient growth versus time for circular pipe flow. (*a*) $\alpha = 0$, $n = 1, 2, 3, 4$, $R = 3000$; (*b*) close-up of (*a*) for small times; (*c*) $\alpha = 0.1$, $n = 1, 2, 3, 4$, $R = 3000$; (*d*) $\alpha = 1$, $n = 1, 2, 3, 4$, $R = 3000$. The solid line denotes $n = 1$, the dashed line $n = 2$, the chain dashed line $n = 3$ and the dotted line $n = 4$. For (*a*) and (*b*) the scaling of the growth function G by the square of the Reynolds number has been used which results in growth curves that are solely dependent on the azimuthal wavenumber n. In (*c*) and (*d*) the streamwise wavenumber α is non-zero and the same scaling does not apply.

Schmid & Henningson 1994

Pipe flow, Re=3000

G: maximum energy growth of initial disturbances

5. Bi-orthogonal projection

5. Bi-orthogonal projection

Demonstration done on blackboard

6. Optimal transient growth

Consider linearised NS system written in this way,

$$\frac{\mathrm{d}q}{\mathrm{d}t} = Aq(t)$$

Equation has general solution, $q(t) = \mathrm{e}^{At} q_o$, where, $\mathrm{e}^{At} = R$

The general solution allows an interpretation in which q_o and $q(t)$ can be considered, respectively, as input and output, connected by the operator, R

We can then ask, what is the input that will lead to the largest output; in other words, what is the most dangerous initial perturbation in terms of energy growth.

The matrix operator connecting input and output can be decomposed via singular-value decomposition (svd),

$$R = U\Sigma V^H$$

where U & V are unitary matrices, i.e.,

$$U^H U = V^H V = I$$

-> each matrix has columns forming an orthonormal basis using the standard Euclidean inner product,

Σ is a diagonal matrix of real, positive entries arranged in descending order.

We thus have,

$$q(t) = U\Sigma V^H q_o,$$
$$U^H q(t) = \Sigma V^H q_o$$

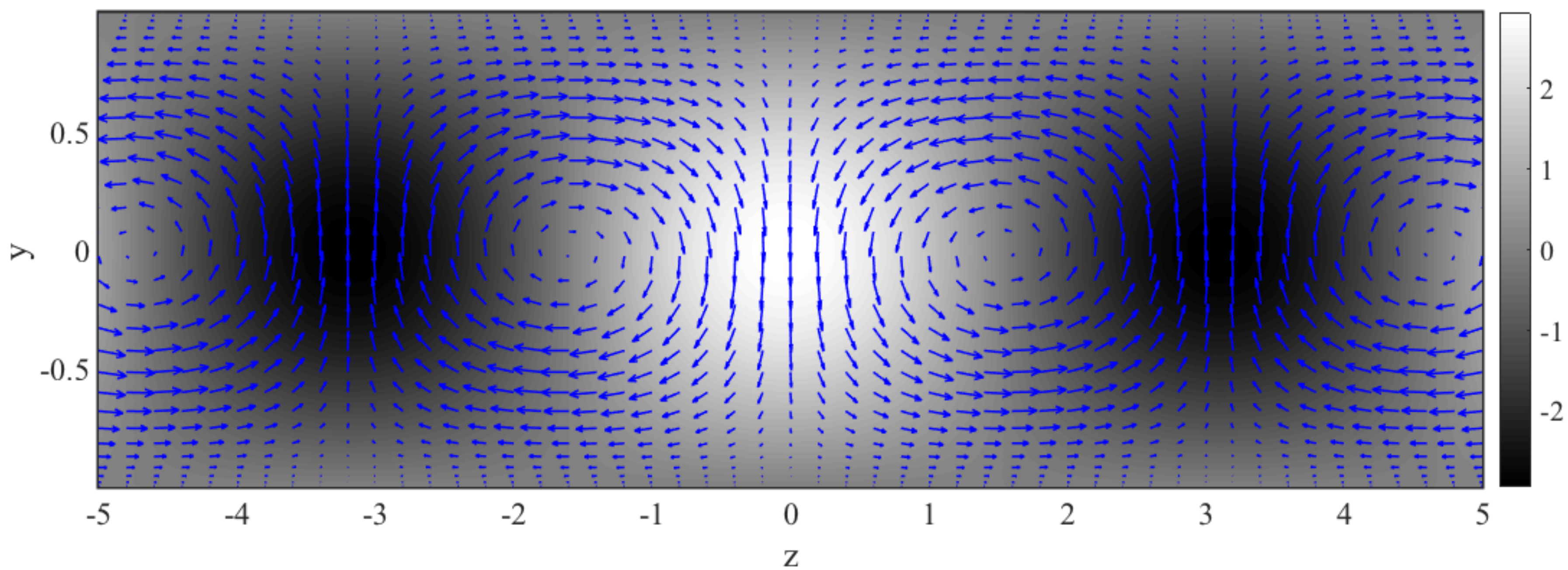
$$q(t) = U\Sigma V^H q_o,$$
$$U^H q(t) = \Sigma V^H q_o$$

The projection of response $q(t)$ onto the modes of the basis U is equal to the projection of the initial condition q_o onto the modes of the basis V multiplied by the corresponding gains σ_i

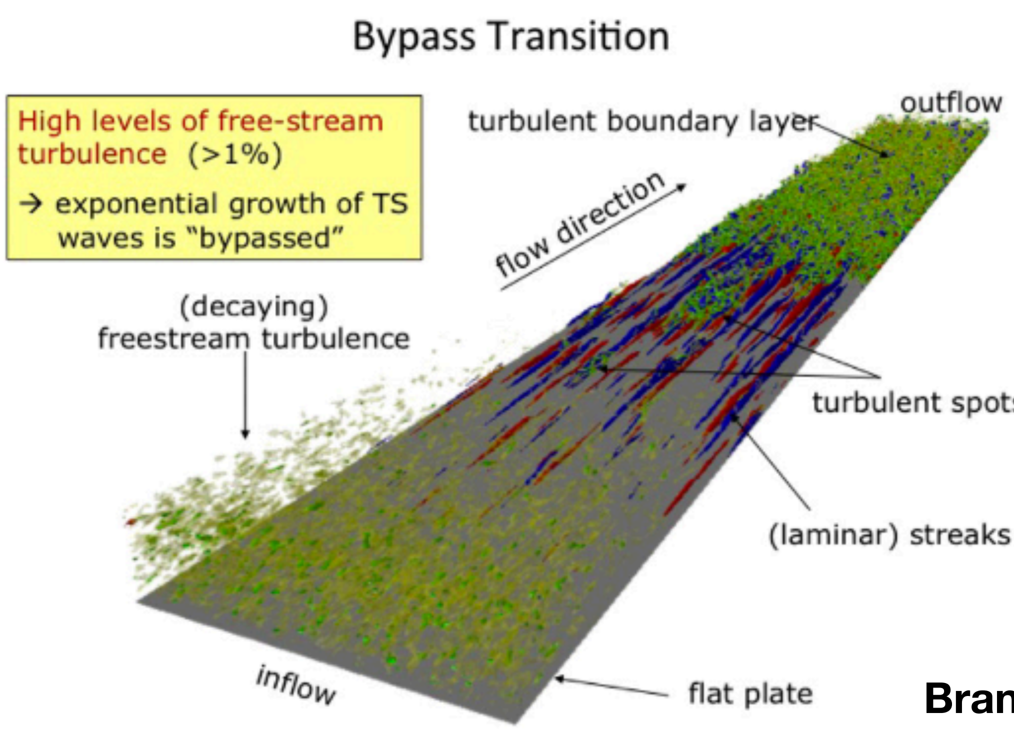
The initial condition that produces the largest response is therefore the first column of V, v_1, and it produces optimal response,

$$u_1 = \sigma_1 v_1$$

Optimal growth in Couette flow - streaks and rolls



By-pass transition in boundary layer



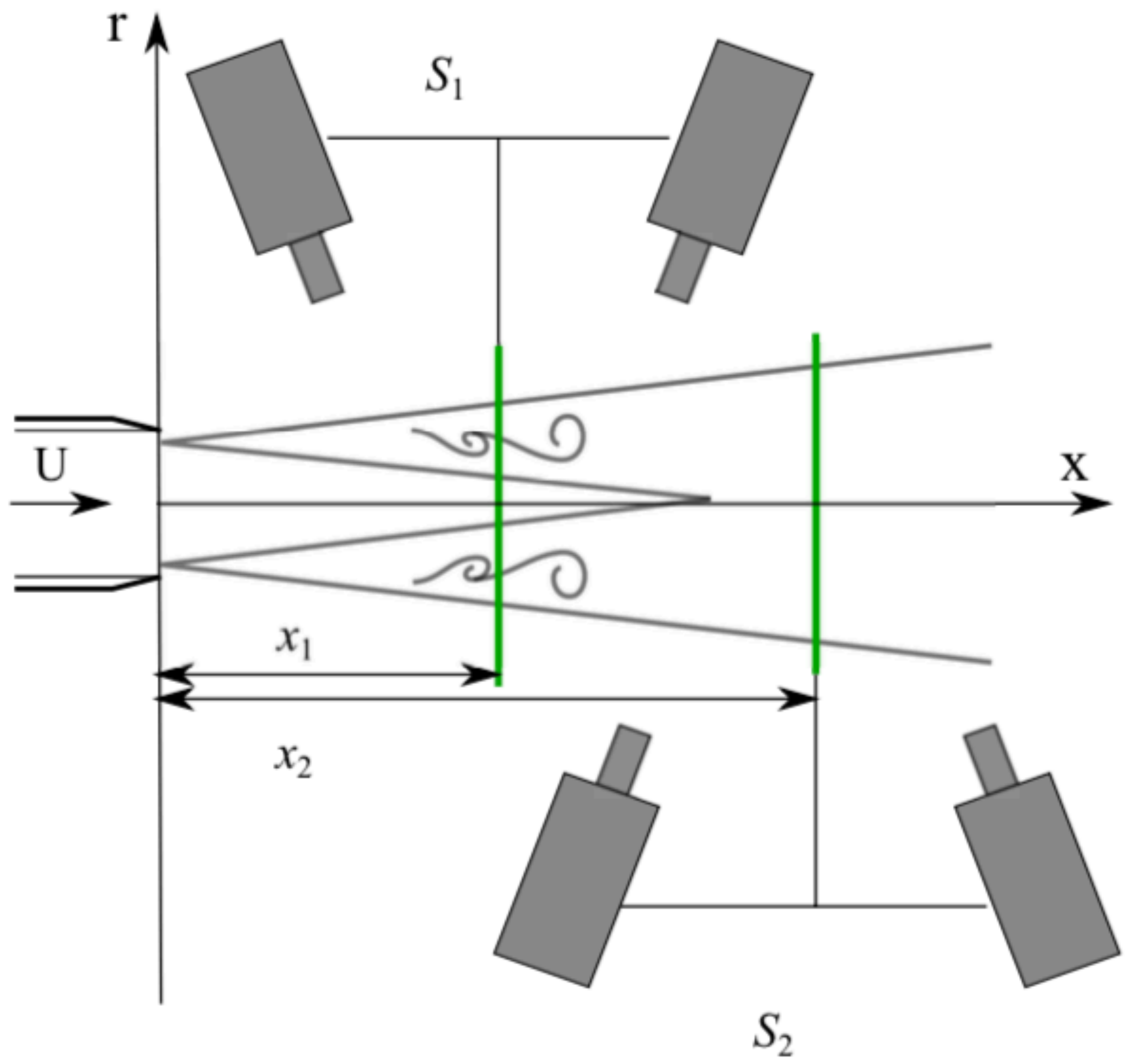
Brandt (2014)

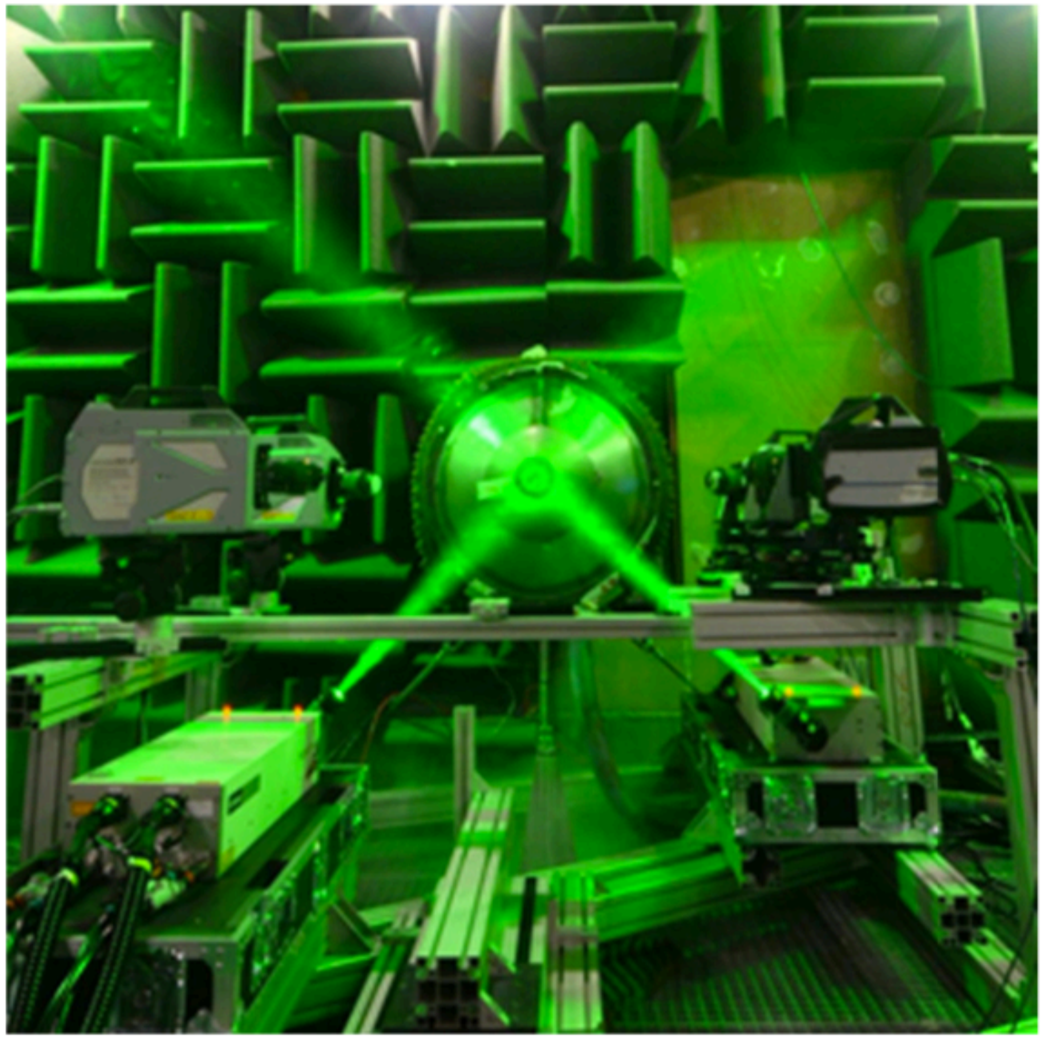
7. Resolvent analysis

Consider linearised NS system written in this way,

$$
\begin{aligned}
(i\omega I - A)\hat{q}(\omega) &= \hat{f}(\omega) \\
L\hat{q}(\omega) &= \hat{f}(\omega) \\
\hat{q}(\omega) &= L^{-1}\hat{f}(\omega) \\
\hat{q}(\omega) &= R\hat{f}(\omega) \\
\hat{q}(\omega) &= U\Sigma V^{H}\hat{f}(\omega)
\end{aligned}
$$

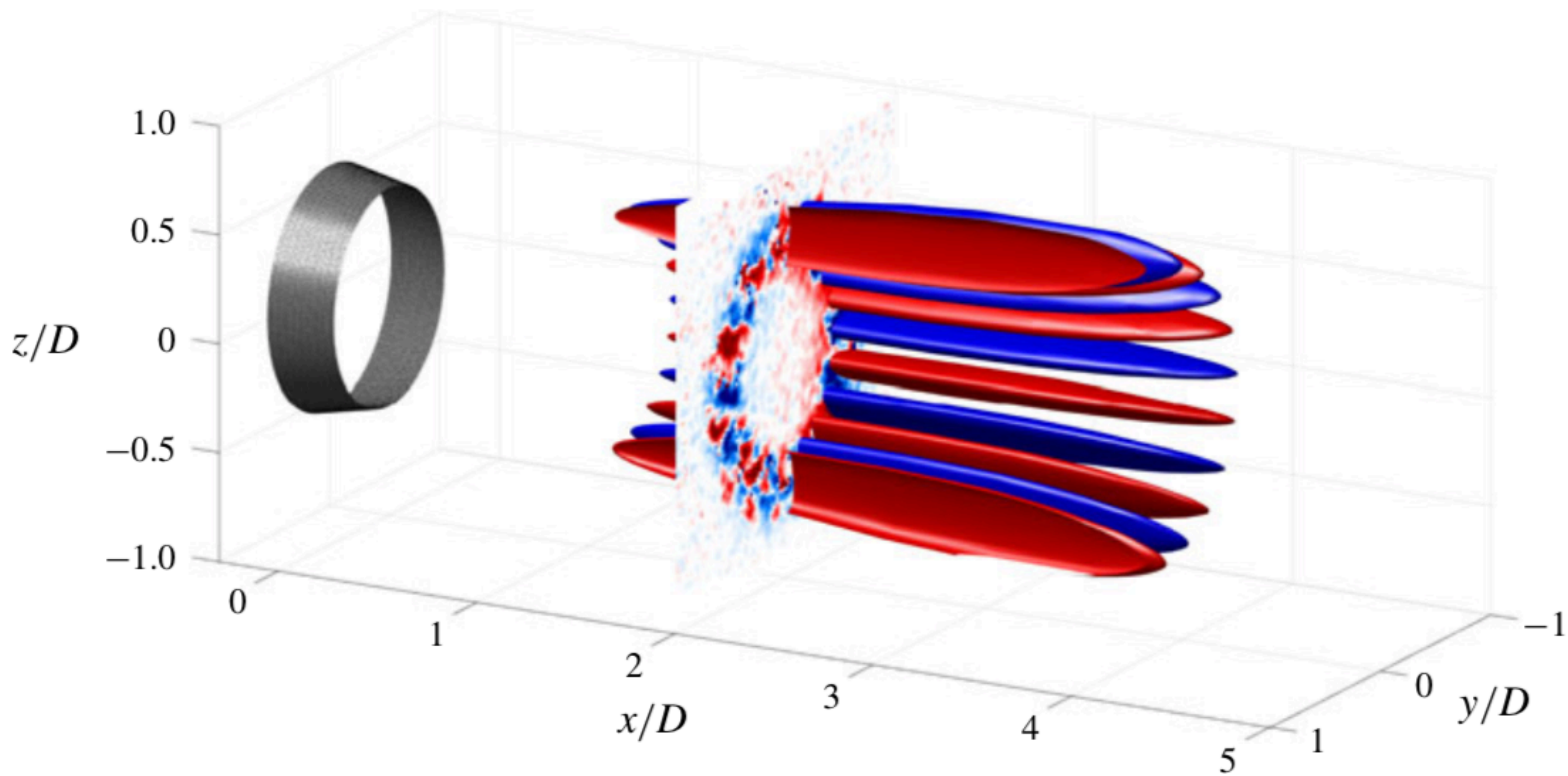
Optimal perturbations in frequency domain





Jaunet, Jordan & Cavalieri (2017) PRF

Optimal perturbations in frequency domain: streaks (m>2) in turbulent jets



Nogueria, Cavalieri, Jordan & Jaunet (2019) JFM

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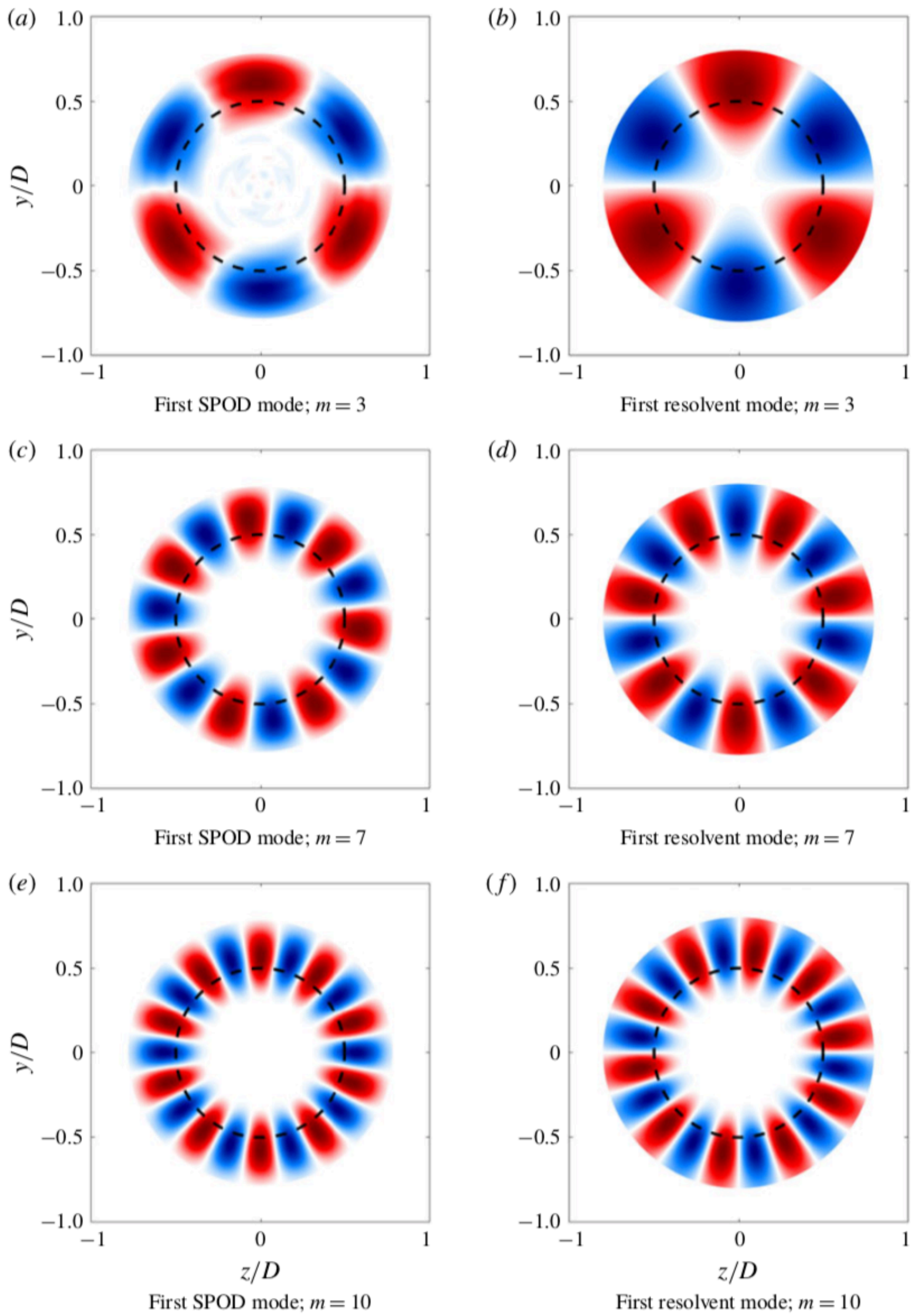
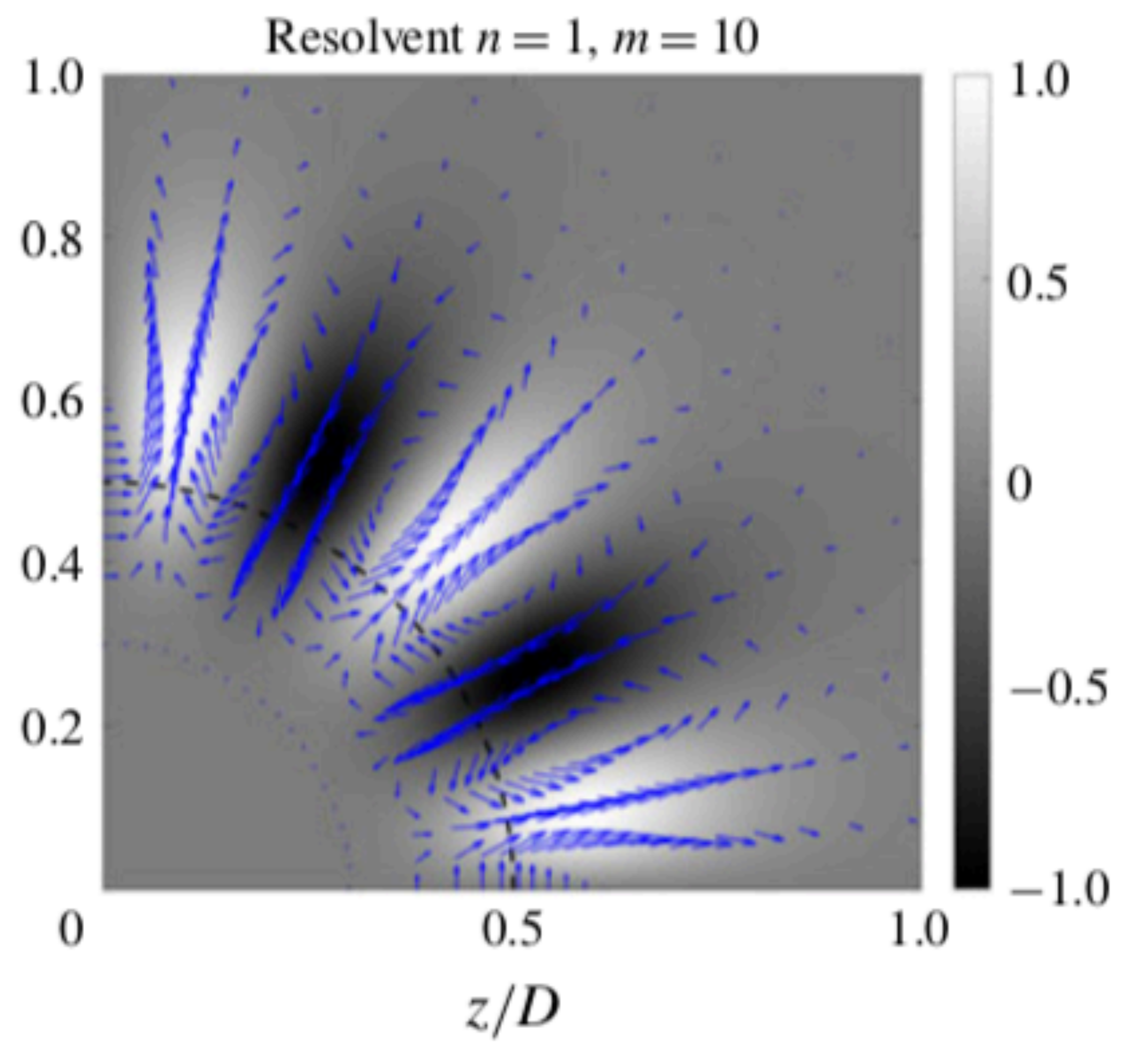
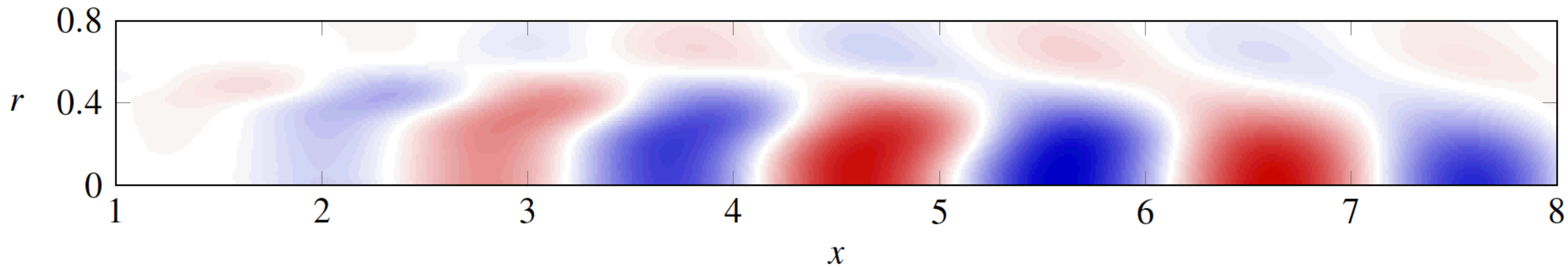




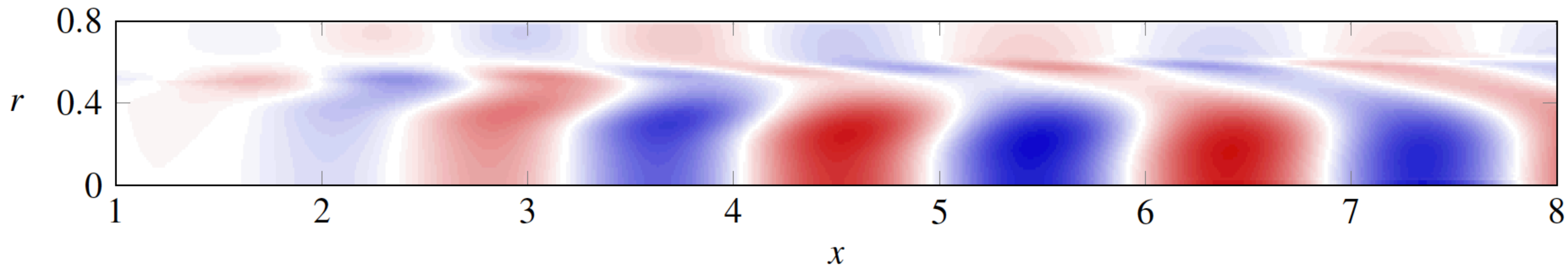
Nogueria, Cavalieri, Jordan & Jaunet (2019) JFM

Optimal perturbations in frequency domain: wavepackets (m=0) in turbulent jets

Experiment



Linear model (Optimal response)



Lesshafft, Semeraro, Jaunet, Cavalieri & Jordan 2019