INSTITUT PPRIME
CNRS–UPR–3346 • UNIVERSITÉ DE POITIERS • ENSMA

DÉPARTEMENT D2 – FLUIDES
THERMIQUE ET COMBUSTION

An introduction to hydrodynamic stability

Lecture 4: Viscous instability

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1. A quick recap. of lecture 3

2. Viscosity and stability of plane Poiseuille flow

3. Orr-Sommerfeld solution for mixing layer

4. Instability in boundary layers

    - Balsius
    - Falkner-Skan

5. Rayleigh's inflection-point theorem

1. Recap. of lecture 3

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Representing a differential operator as a matrix

Turning a differential equation into a matrix eigenvalue problem that can be solved in Matlab with eig(L,F)

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}\right]\mathbf{v} = \lambda(-1)\mathbf{v}$$

$$\mathrm{L}\mathbf{v} = c\mathrm{F}\mathbf{v}$$

$$\left[U\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2\right) - \frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\right]\mathbf{v} = c\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2\right]\mathbf{v}$$

1. Recap. of lecture 3

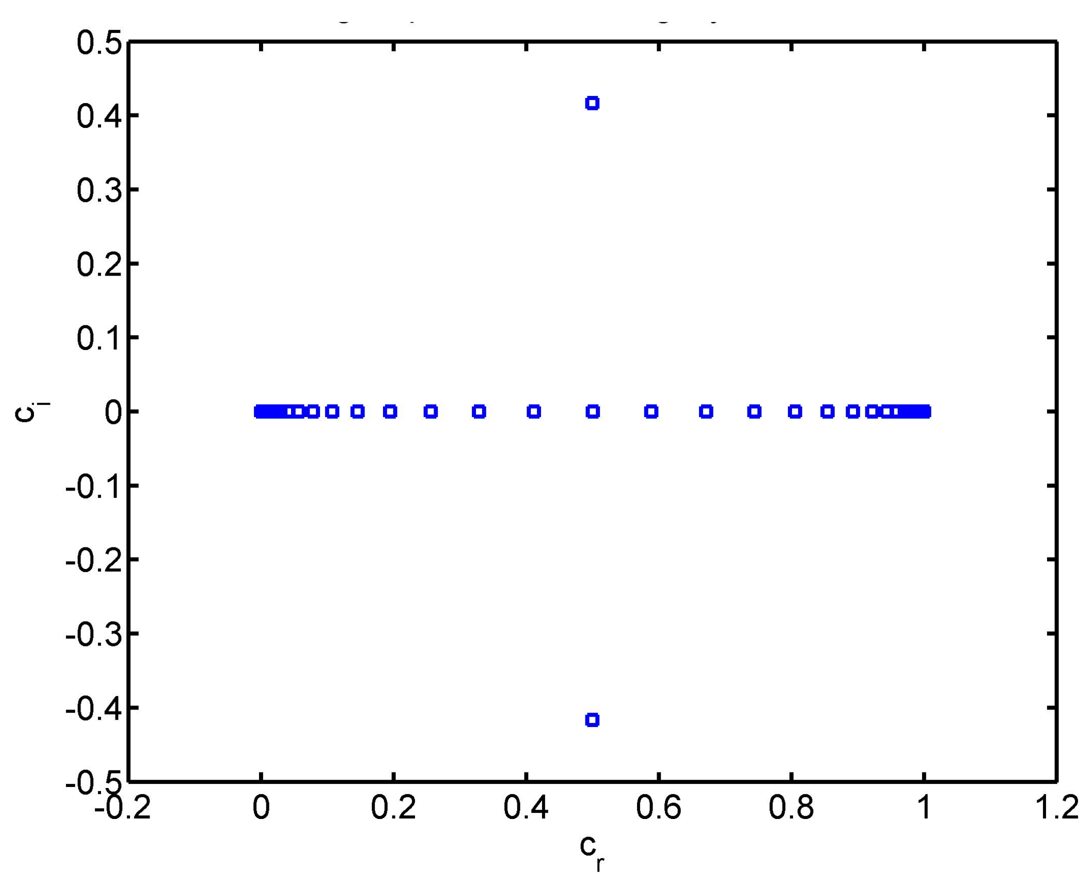
Can use finite-difference methods or…

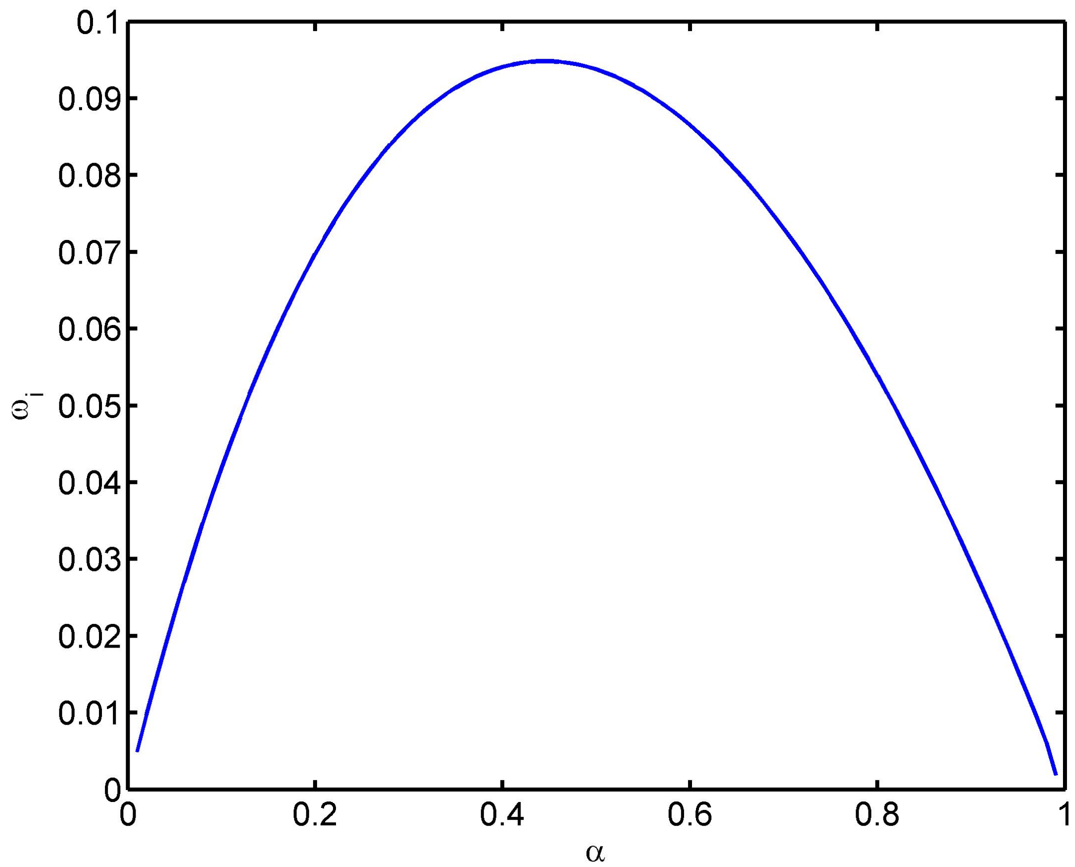
Algebraic (polynomial) interpolants: appropriate for bounded and/or non-periodic domains, as opposed to trigonometric interpolants, suitable for periodic domains.

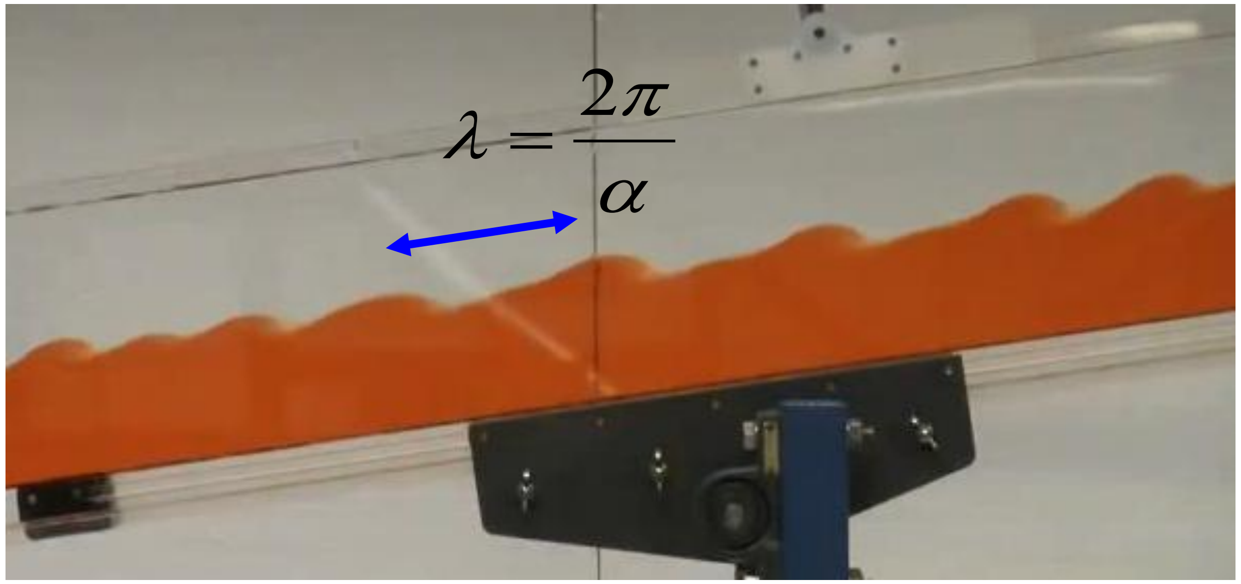
Procedure:

- Use an Nth-order polynomial to represent the discrete data,
- Derivative of discrete data expressed, at each grid point, in terms of (analytical) derivatives of the interpolant,
- This provides a set of polynomial coefficients for each grid point
- The derivatives at each grid point is a function of all other grid points,
- Differentiation can be expressed in matrix form; the matrix is full on account of previous point

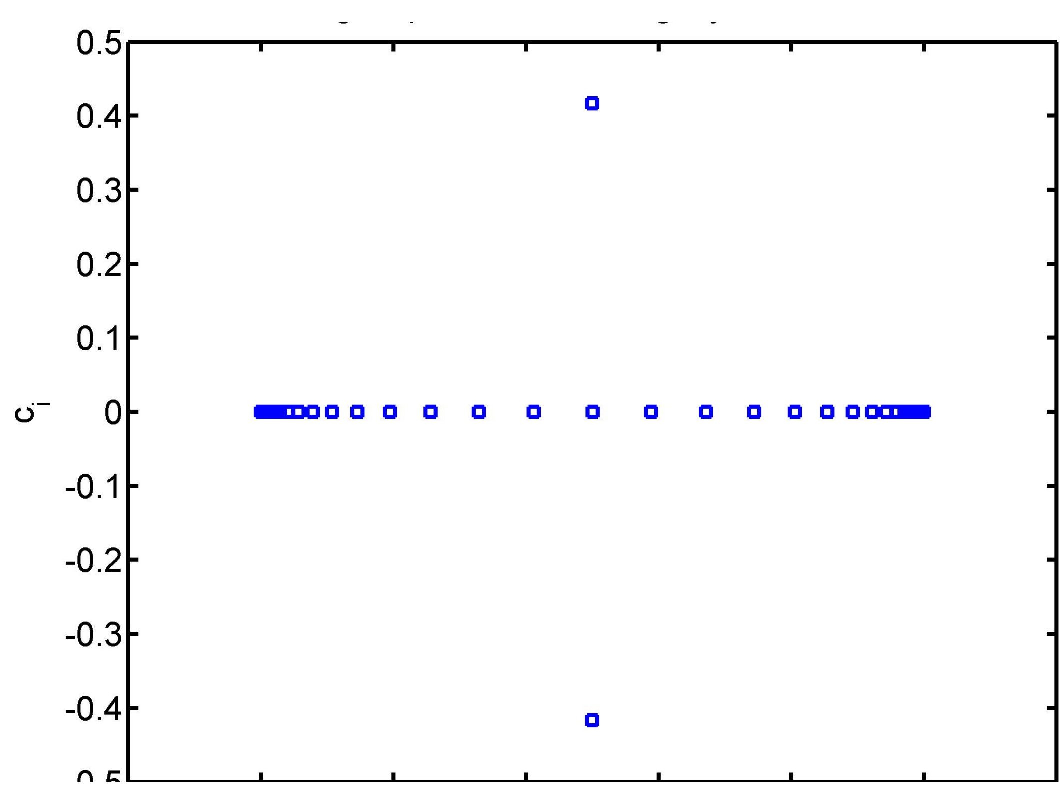
1. Recap. of lecture 3

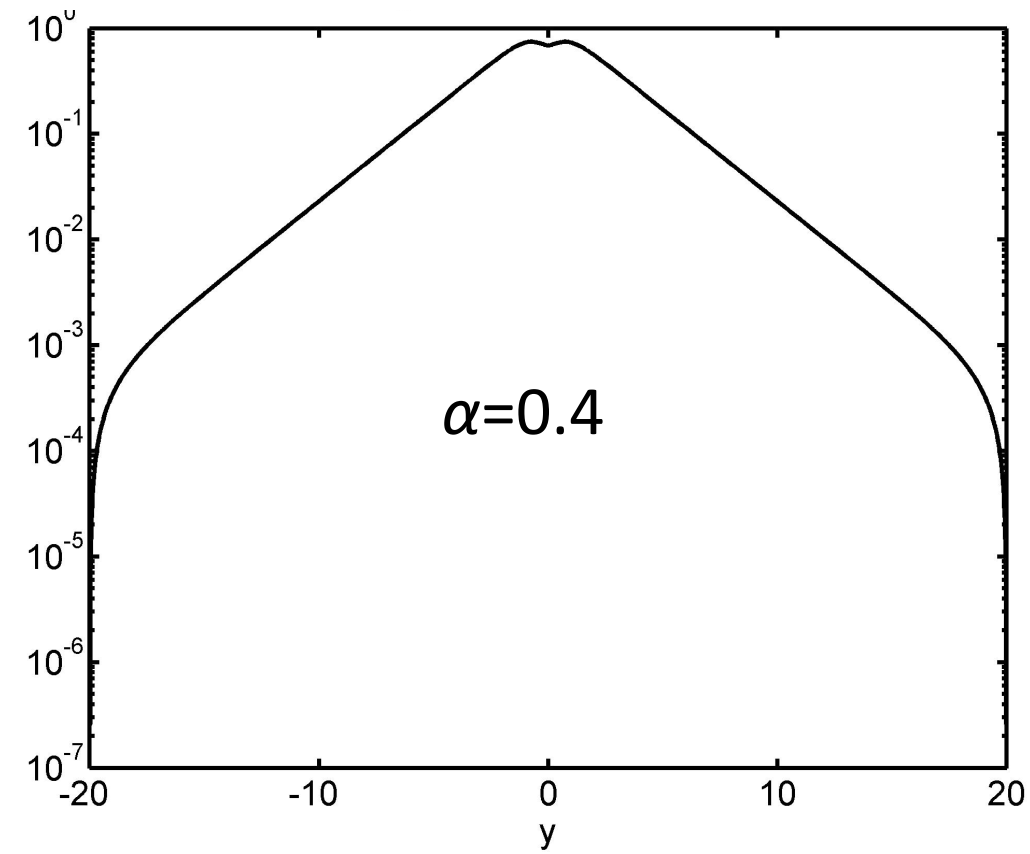


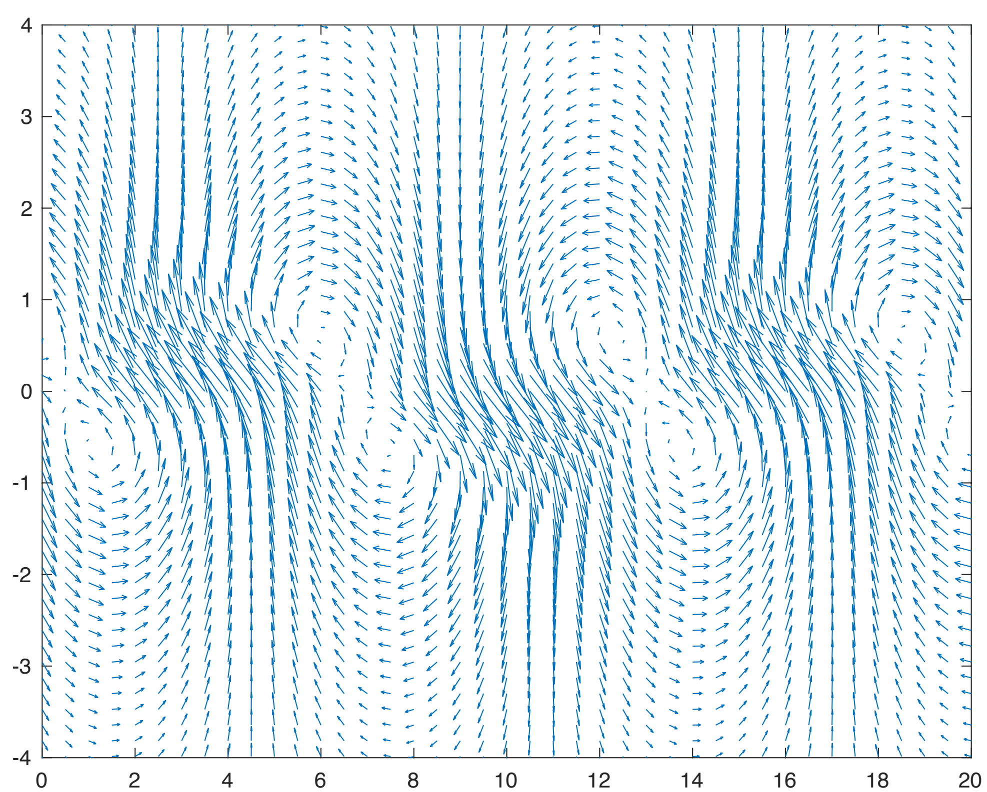


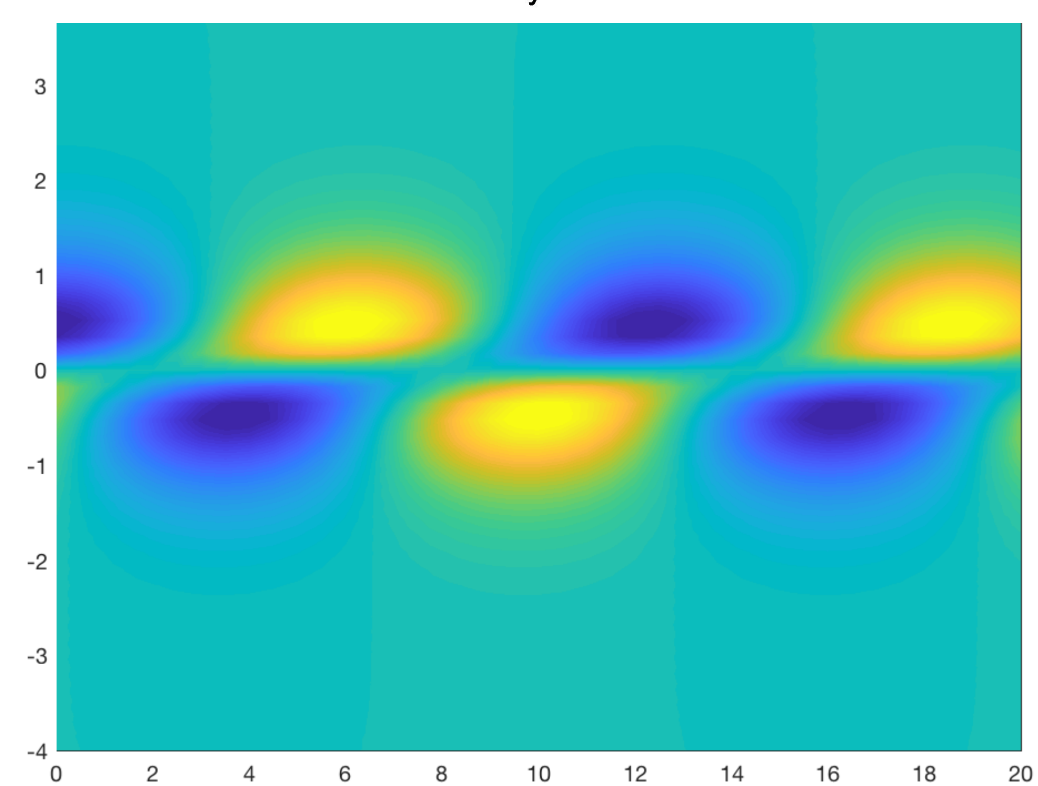


1. Recap. of lecture 3



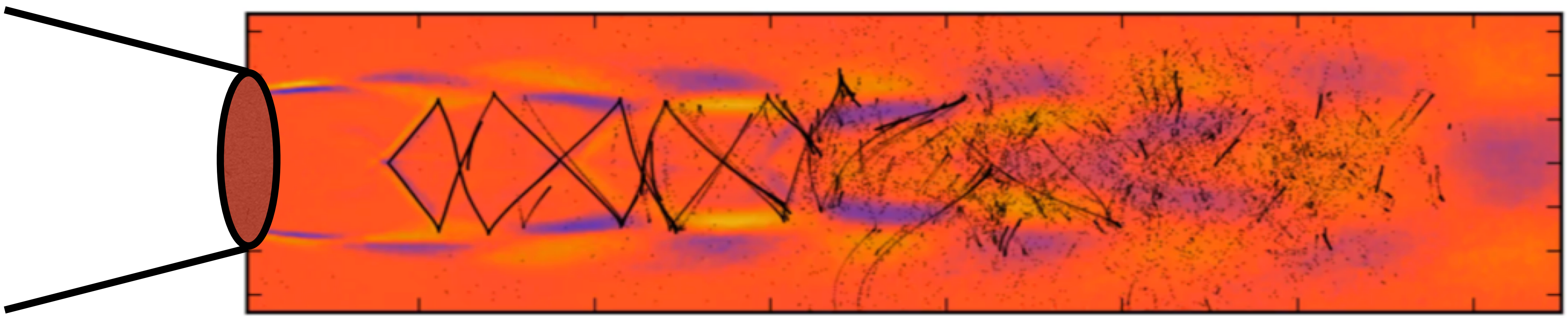


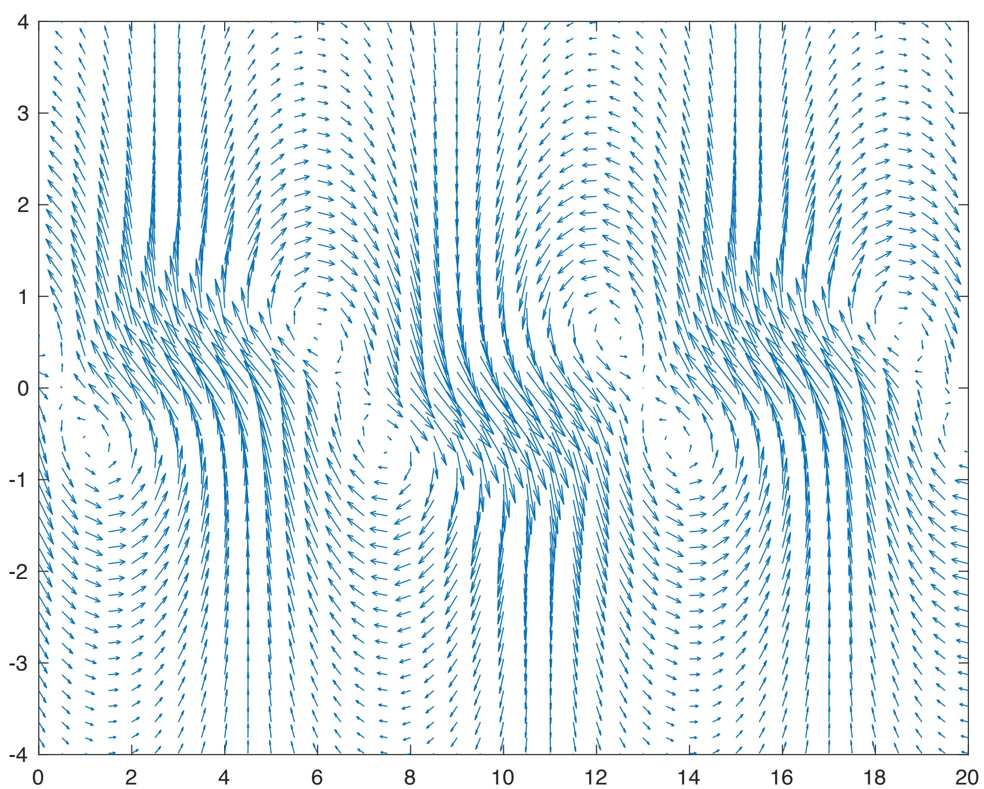


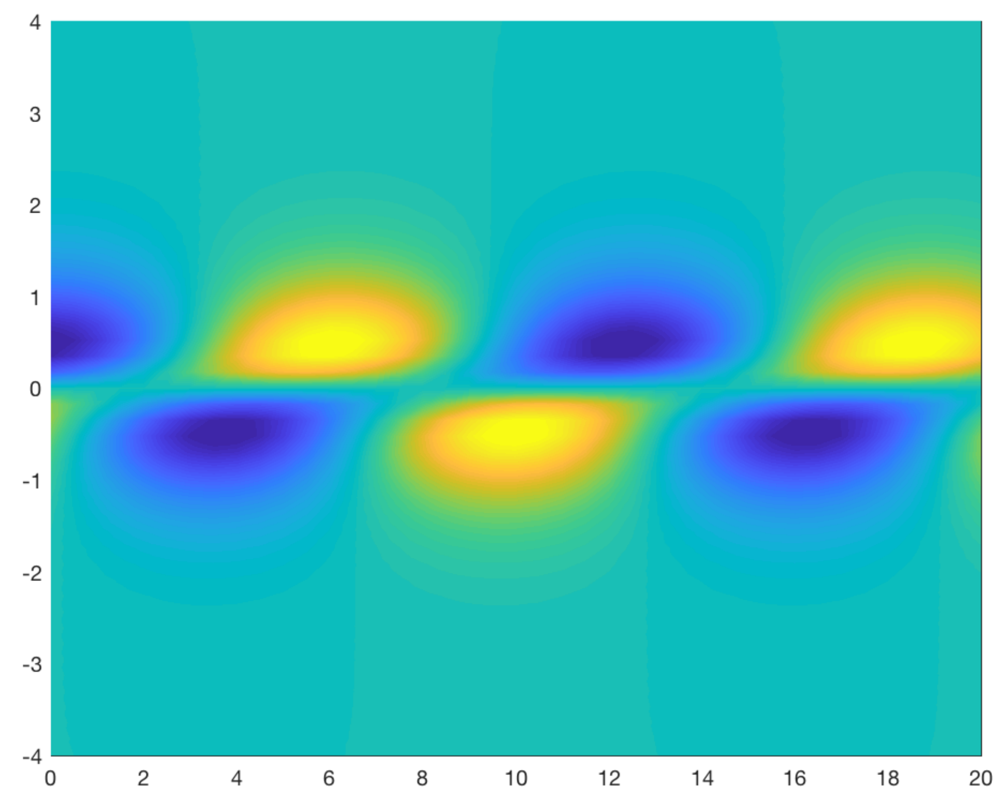


1. Recap. of lecture 3

Edgington-Mitchell et al. 2017

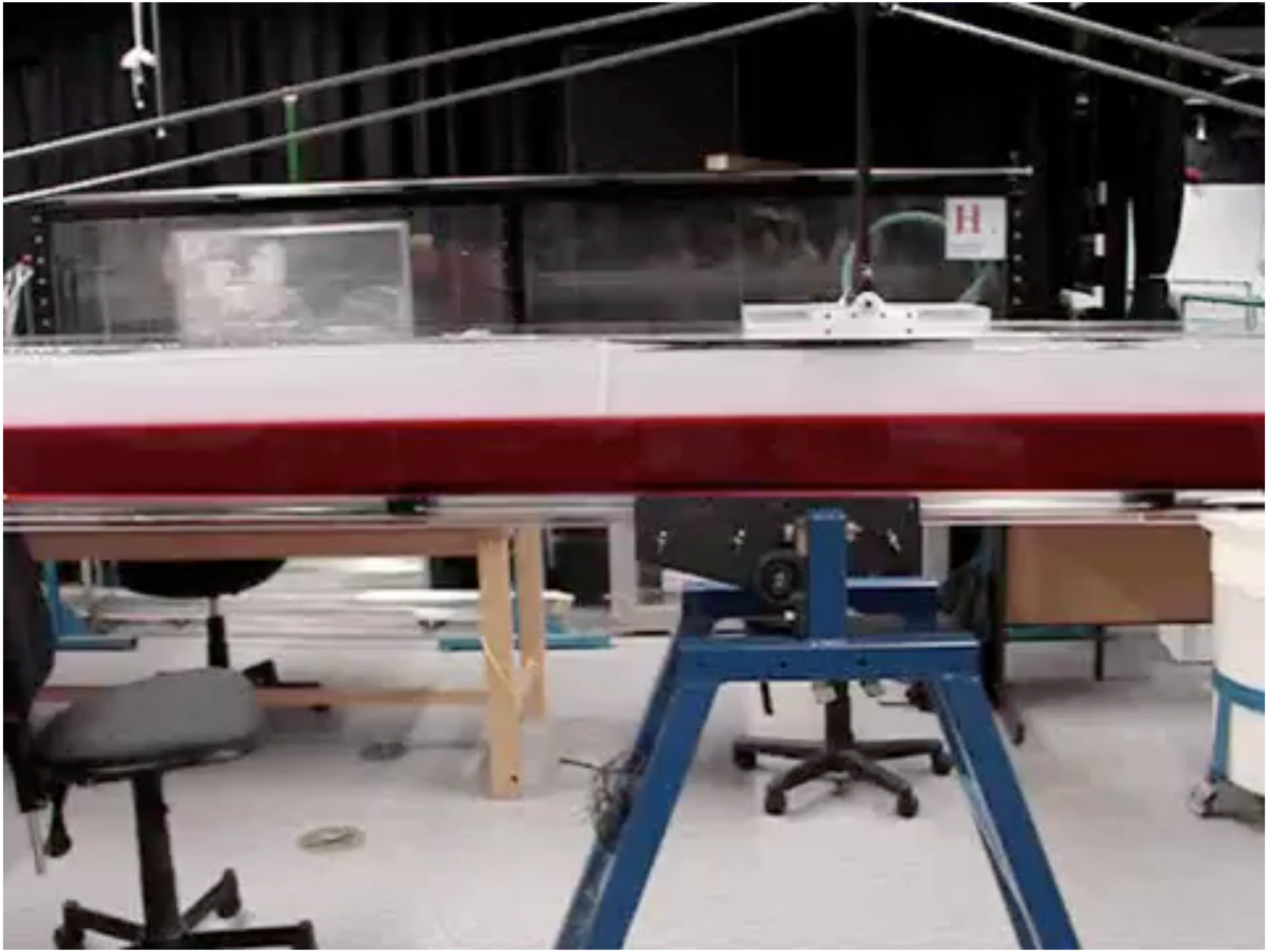






2. Viscosity and stability of plane Poiseuille flow

2. Viscosity and the stability of Poiseuille flow



2. Viscosity and the stability of Poiseuille flow

Orr-Sommerfeld equation

$$(U-c)\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right]\hat{v}-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\hat{v}=\frac{1}{i\alpha\mathrm{Re}}\left[\frac{\mathrm{d}^4}{\mathrm{d}y^4}-2\alpha^2\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\alpha^4\right]\hat{v}$$

Rearrange to take form of eigenvalue problem

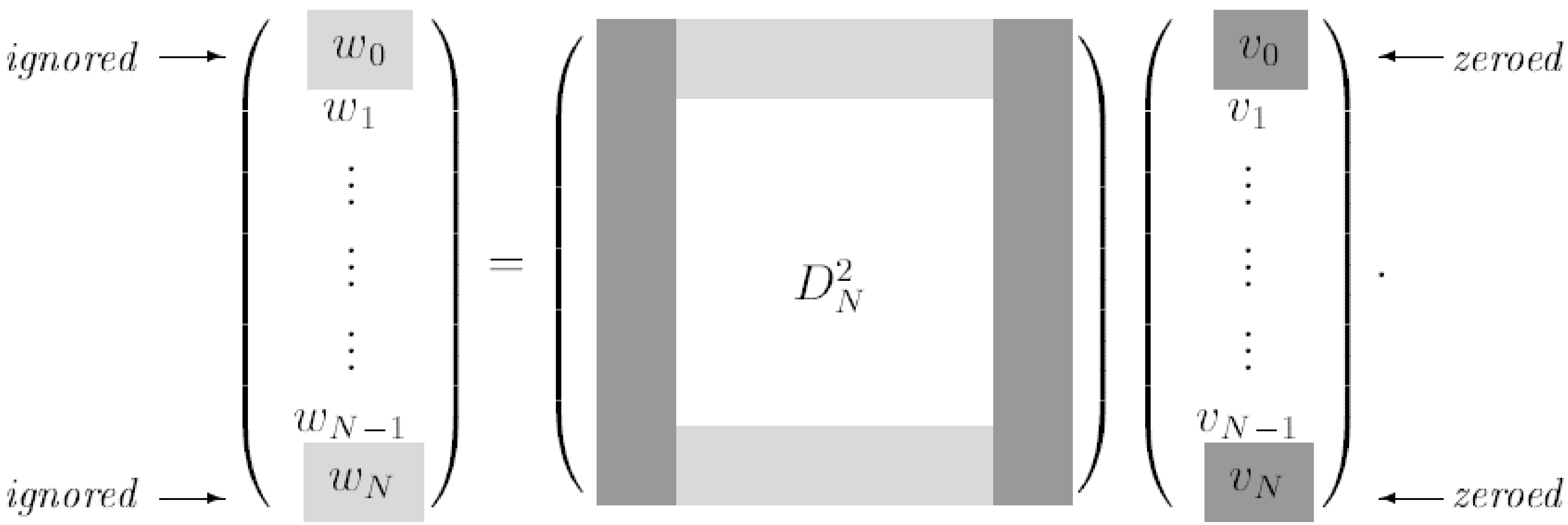
$$\left[U\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right)-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}-\frac{1}{i\alpha\mathrm{Re}}\left(\frac{\mathrm{d}^4}{\mathrm{d}y^4}-2\alpha^2\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\alpha^4\right)\right]\hat{v}=c\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right]\hat{v}$$

$$\mathbf{L}\mathbf{v}=c\mathbf{F}\mathbf{v}$$

Exercise: write a code to solve the temporal stability problem for the Orr-Sommerfeld equation for plane Poiseuille flow (U=1-y2), with BCs:

$$v(+1)=\frac{dv}{dy}(+1)=v(-1)=\frac{dv}{dy}(-1)=0$$

More numerical tricks: another way to impose homogeneous Dirichlet boundary conditions



L=L(2:N,2:N)
F=F(2:N,2:N)

Because the boundary values have been specified, the number of degrees of freedom has been reduced by 2: the matrix must be reduced accordingly to ensure that it not be overconstrained.

More numerical tricks: a way to simultaneously impose homogeneous Dirichlet and Neumann boundary conditions

Introduce a new variable, *q(y)*

$$\mathbf{v}(y) = \left(1 - y^2\right)q(y)$$

$$\frac{d\mathbf{v}}{dy}(y) = (2y)q(y) + \left(1 - y^2\right)q'(y)$$

$$\frac{d^4\mathbf{v}}{dy^4} = \left(1 - y^2\right)\frac{d^4 q}{dy^4} - 8y\frac{d^3 q}{dy^3} - 12\frac{d^2 q}{dy^2}$$

If q(-1)=q(1)=0 **then**

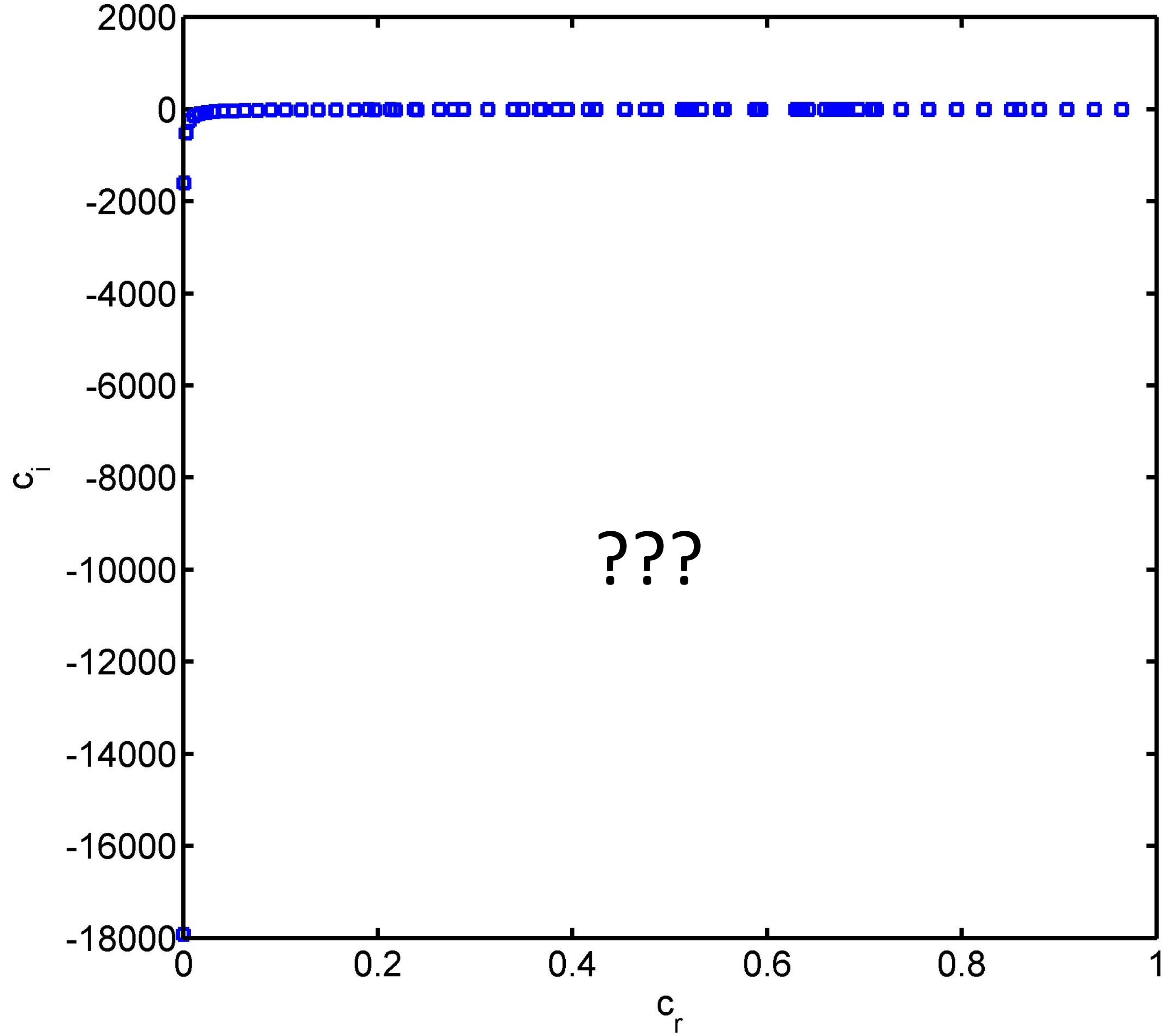
v(-1)=v(1)=0

AND

dv/dy(-1)=dv/dy(1)=0

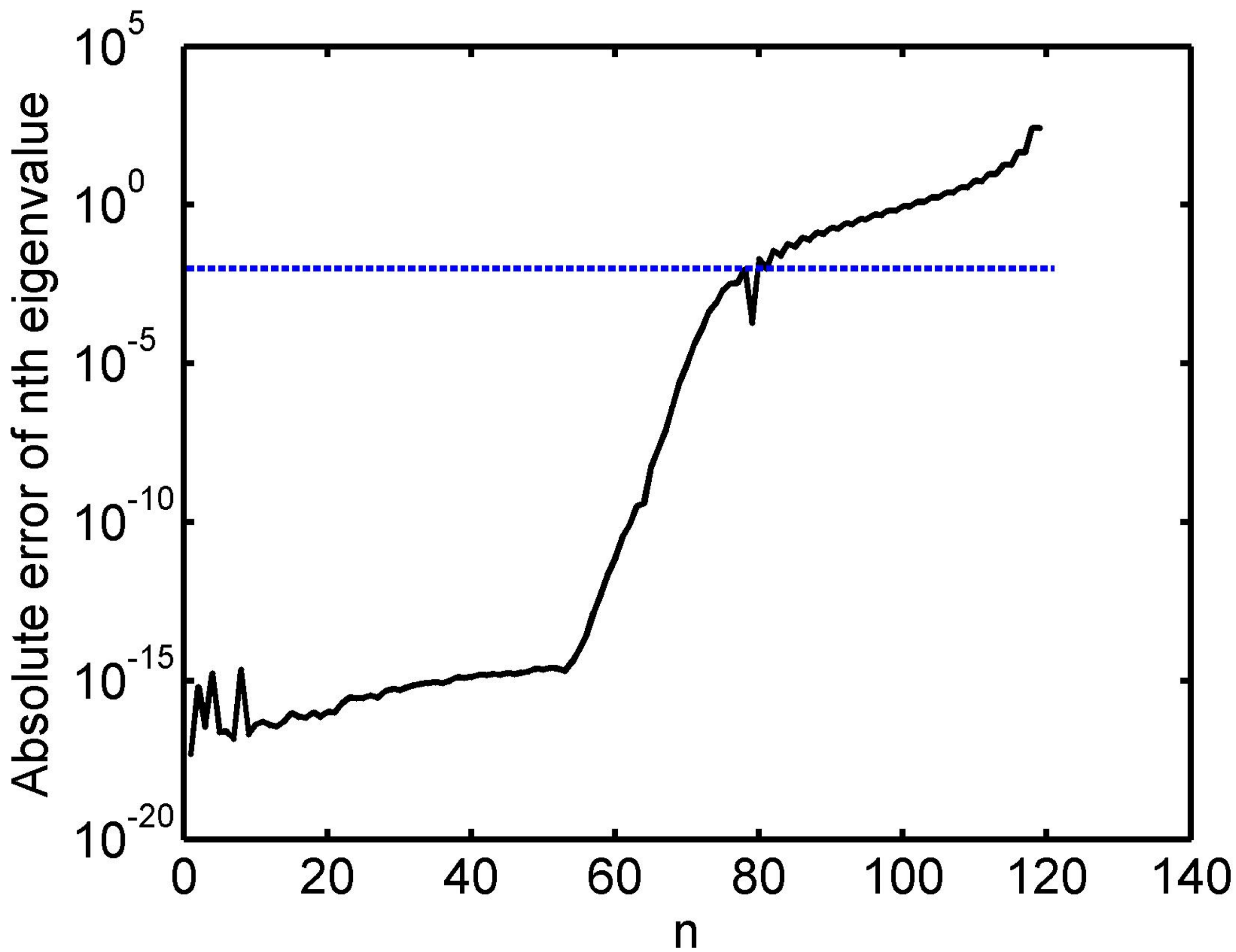
```
D2 = D*D; D2 = D2(2:N,2:N);
S = diag([0; 1./(1-y(2:N).^2); 0]);
D4 = (diag(1-y.^2)*D^4 - 8*diag(y)*D^3 - 12*D^2);
D4=D4(2:N,2:N);
D4=D4*S(2:N,2:N);
y = y(2:N);
D = D(2:N,2:N);
```

Results


2000
0
-2000
-4000
-6000
-8000
-10000
-12000
-14000
-16000
-18000
c_i
0
0.2
0.4
0.6
0.8
1
c_r
???

Remember this? Not all eigenvalues are converged

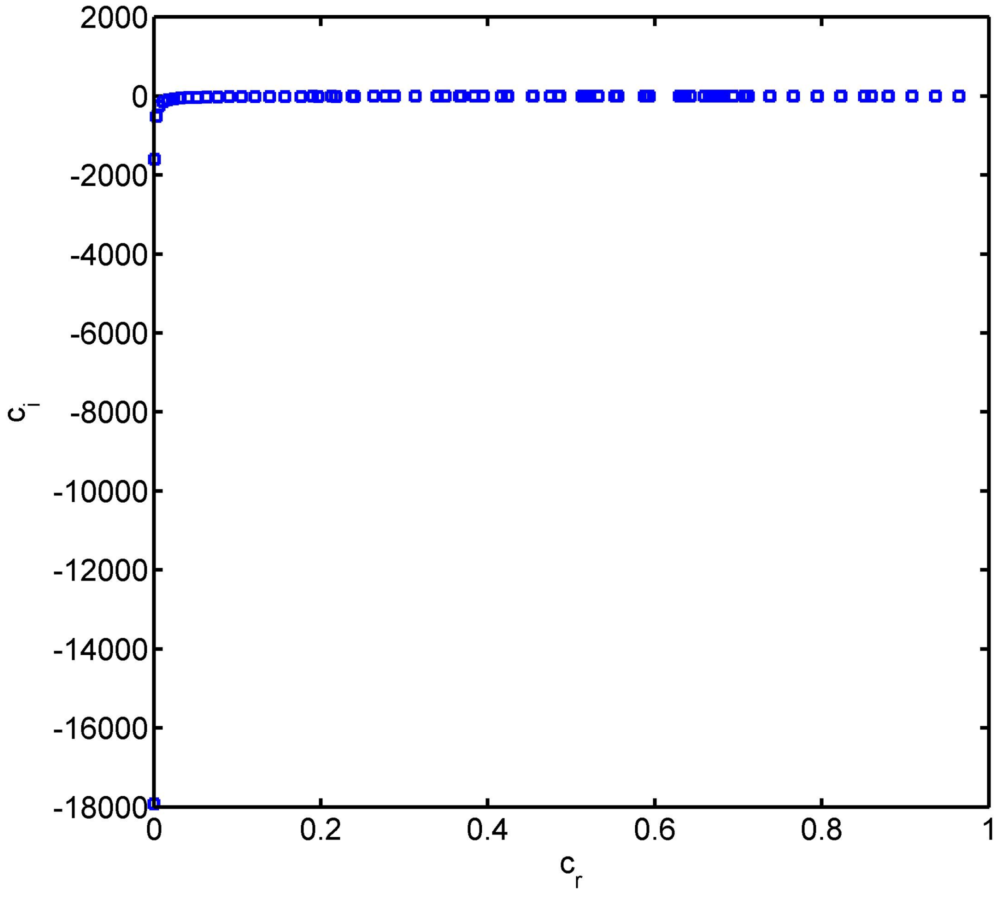
$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}\right]\mathbf{v} = \lambda(-1)\mathbf{v}$$

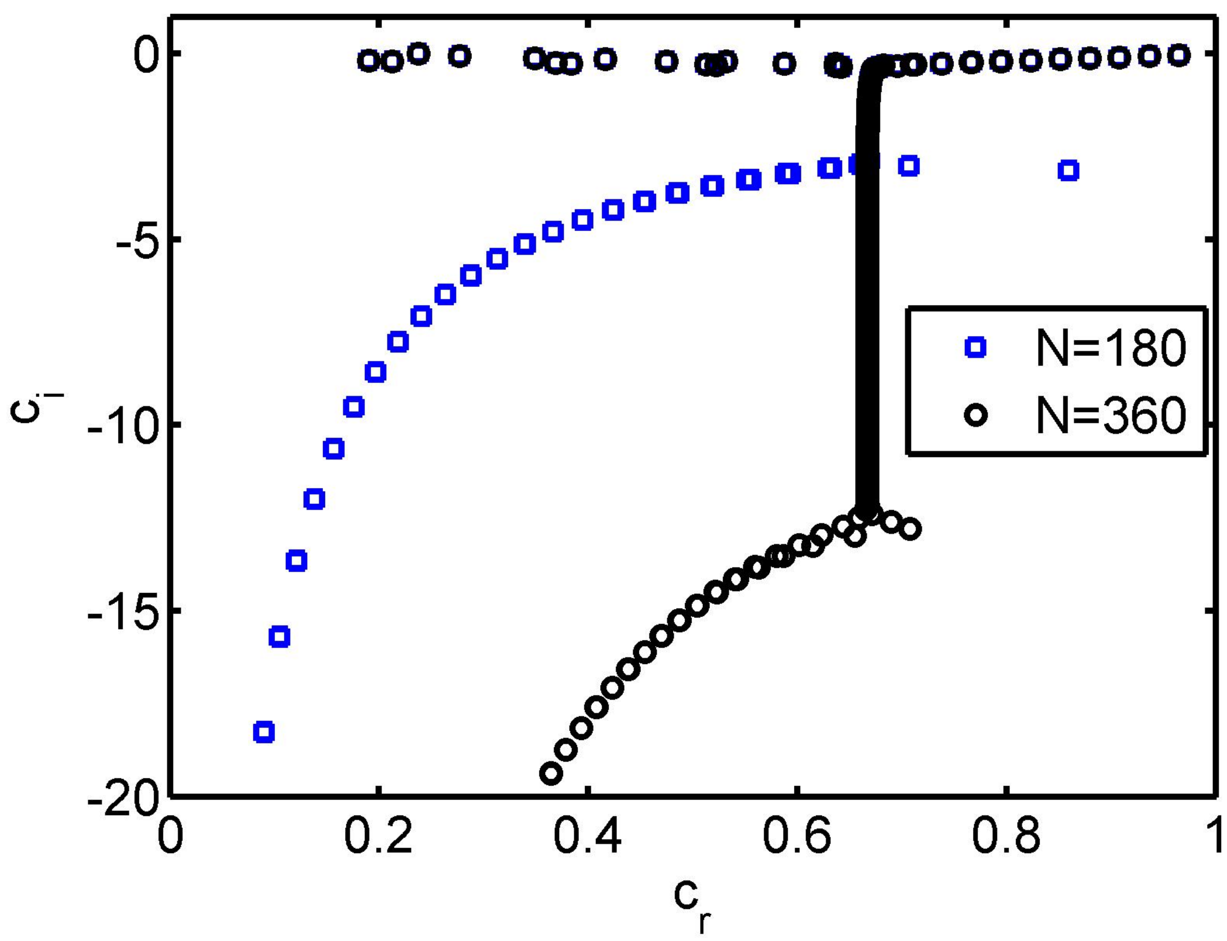


Orr-Sommerfeld:
no analytical solutions

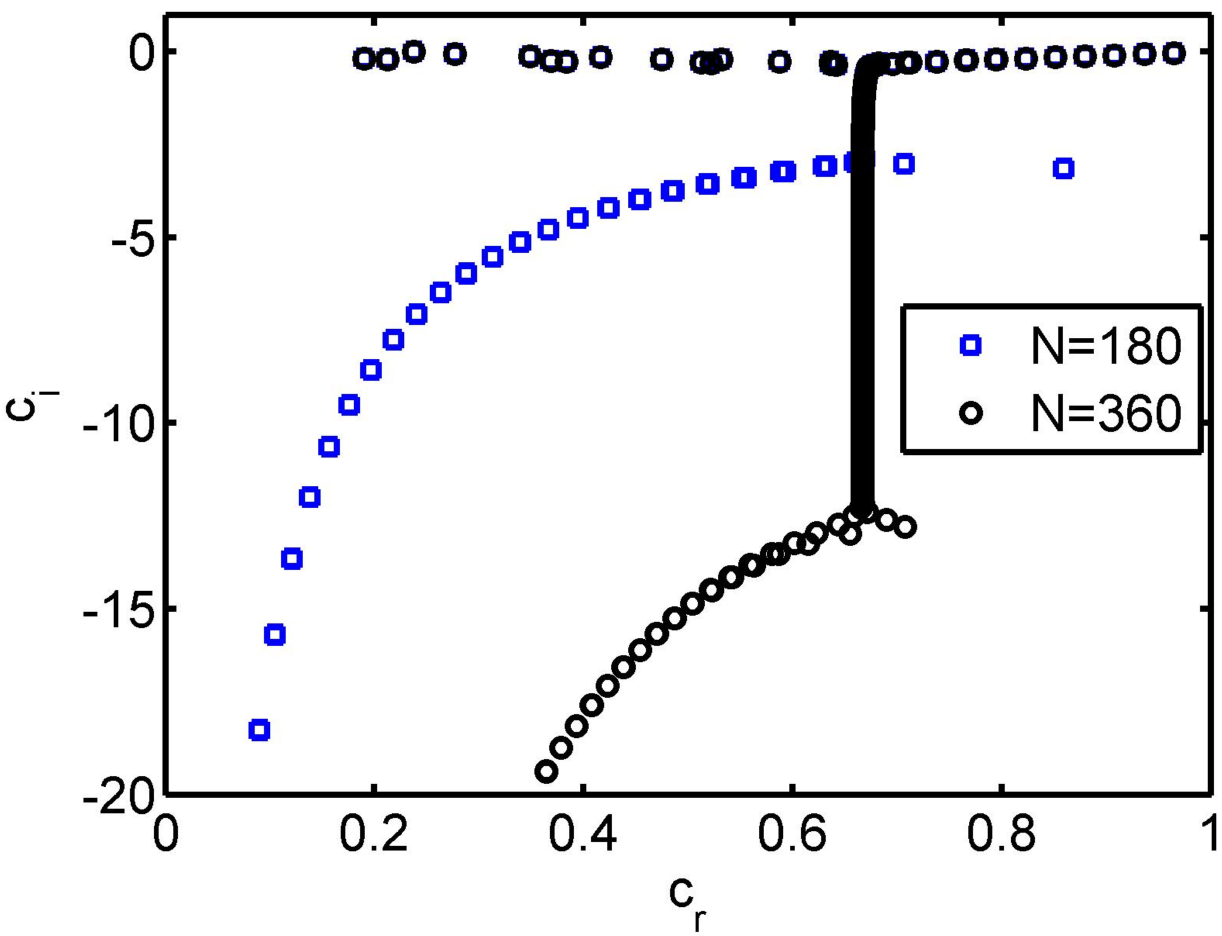
How do we decide which eigenvalues are accurate?

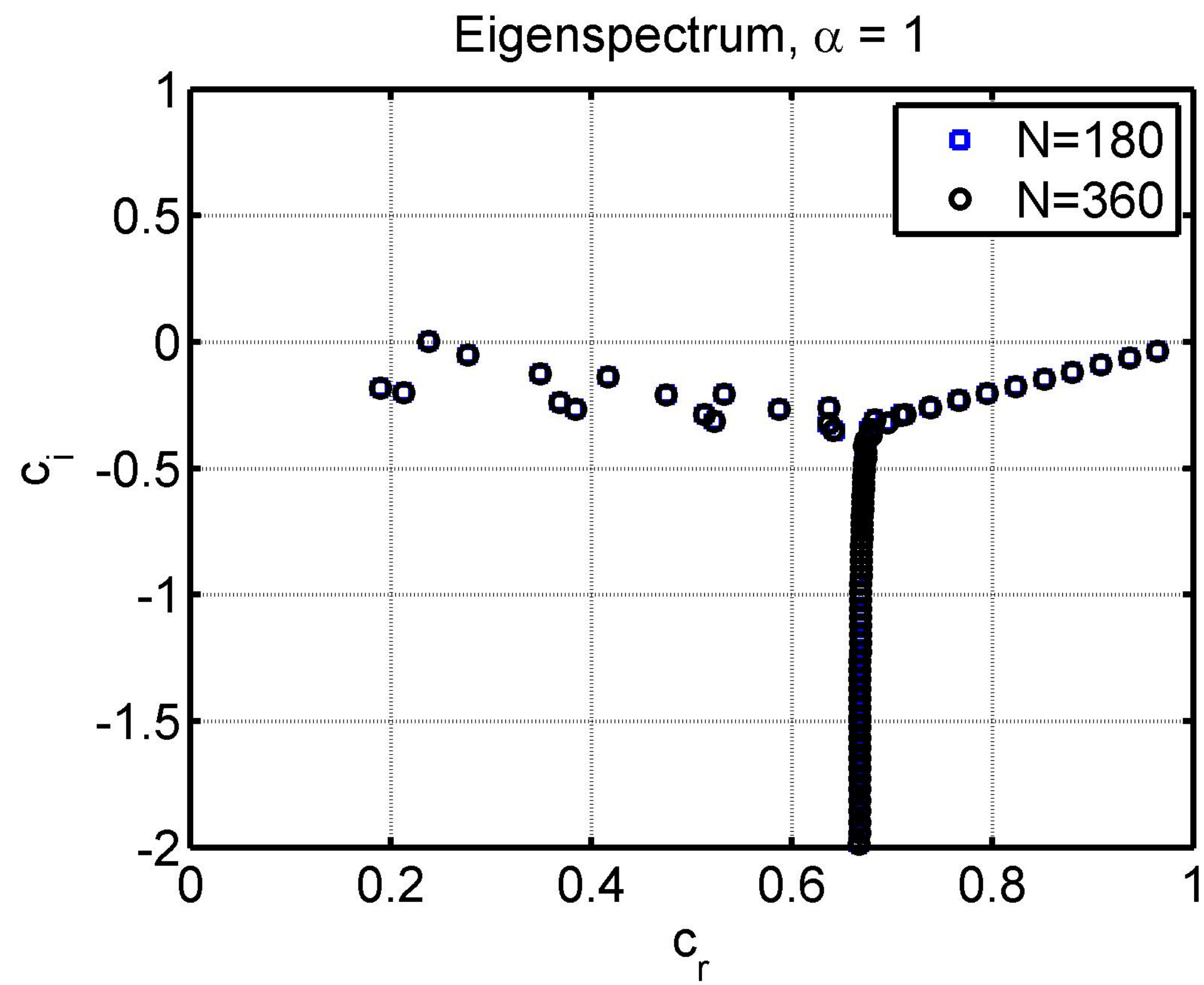
Results


2000
0
-2000
-4000
-6000
-8000
-10000
-12000
-14000
-16000
-18000
c_i
0
0.2
0.4
0.6
0.8
1
c_r

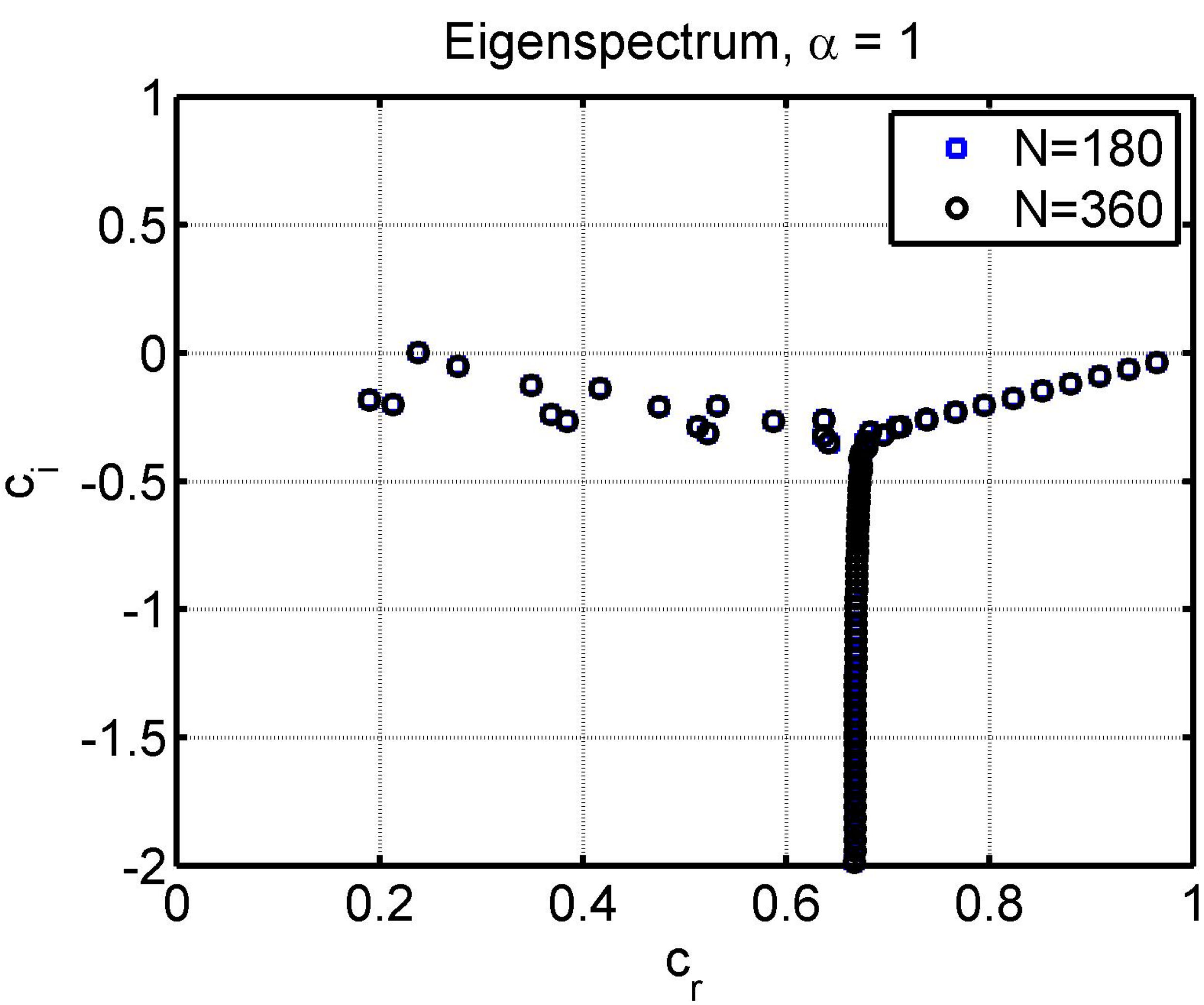

0
-5
-10
-15
-20
c_i
0
0.2
0.4
0.6
0.8
1
c_r
N=180
N=360

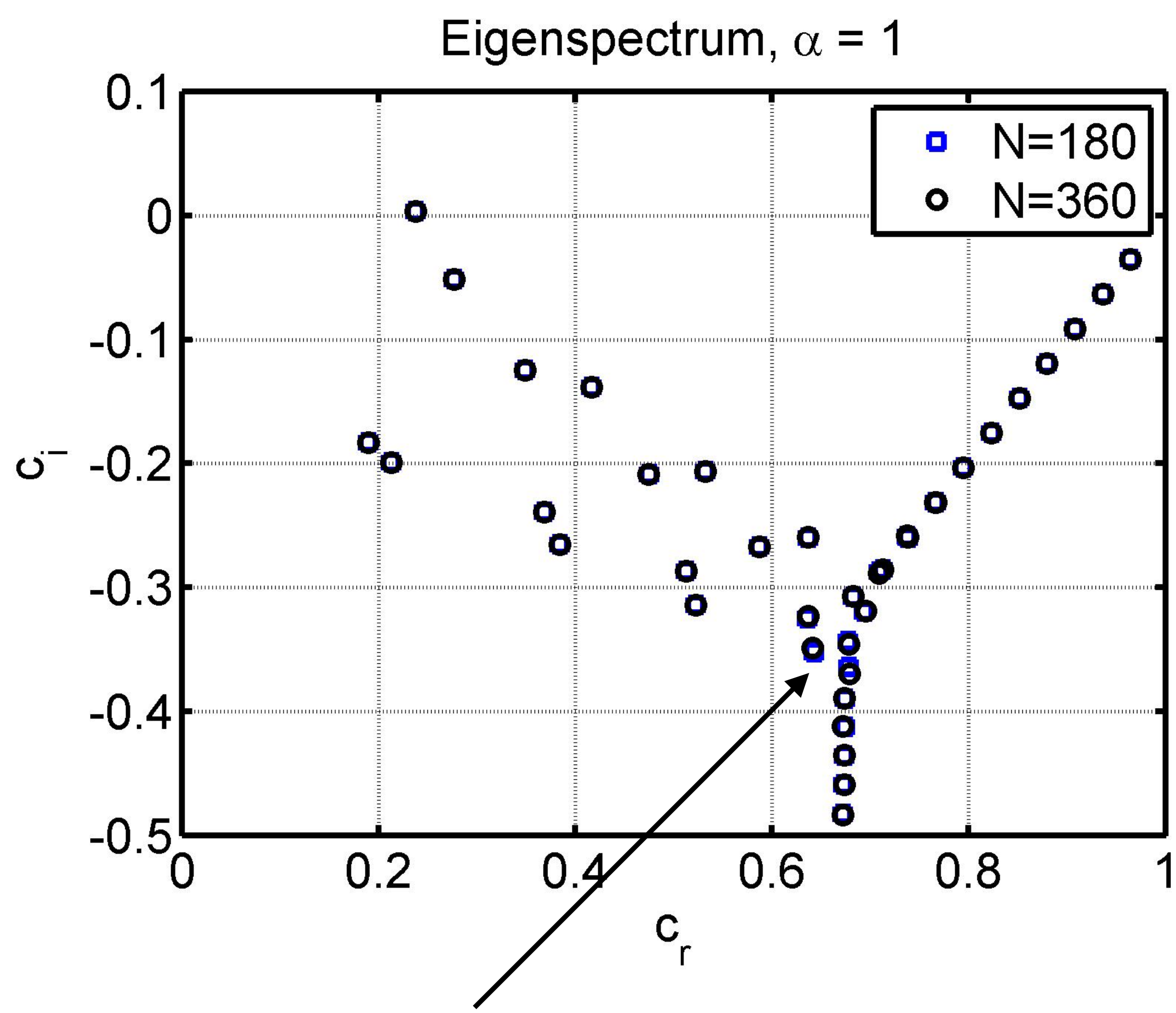
Results


N=180
N=360
c_i
c_r
0
-5
-10
-15
-20
0
0.2
0.4
0.6
0.8
1


Eigenspectrum, α = 1
N=180
N=360
c_i
c_r
1
0.5
0
-0.5
-1
-1.5
-2
0
0.2
0.4
0.6
0.8
1

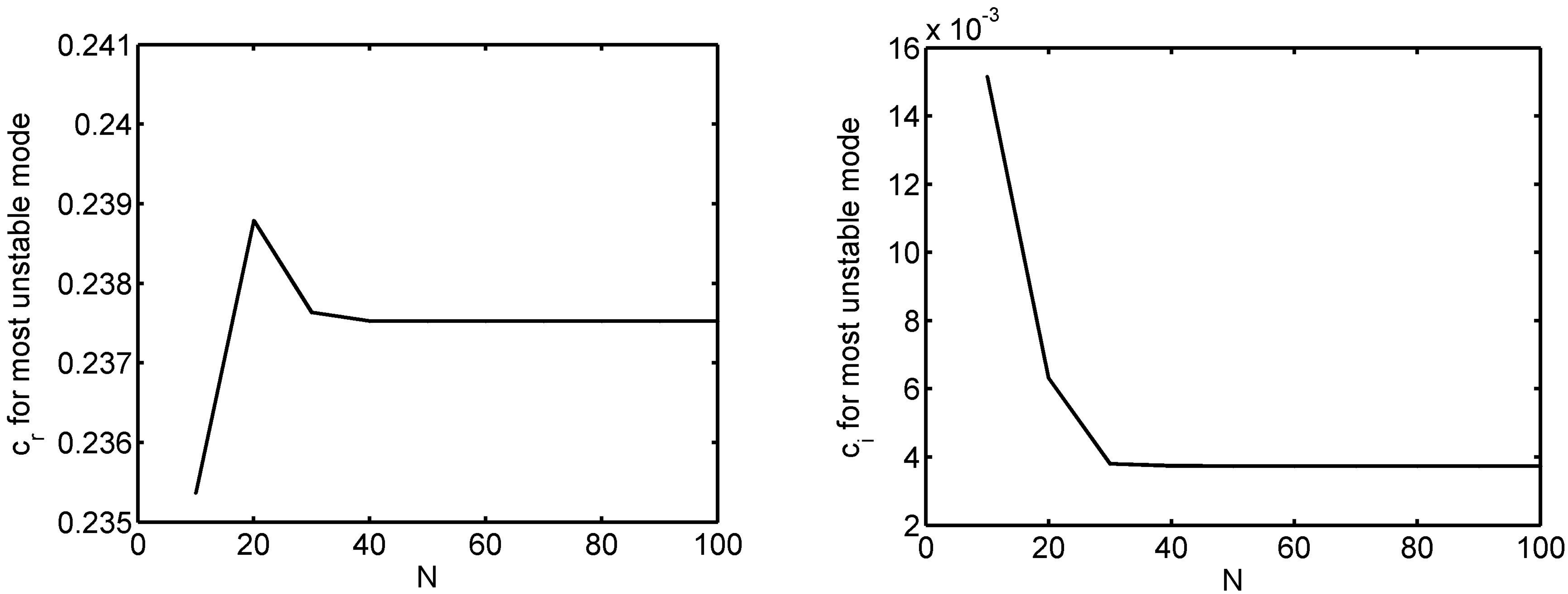
Results





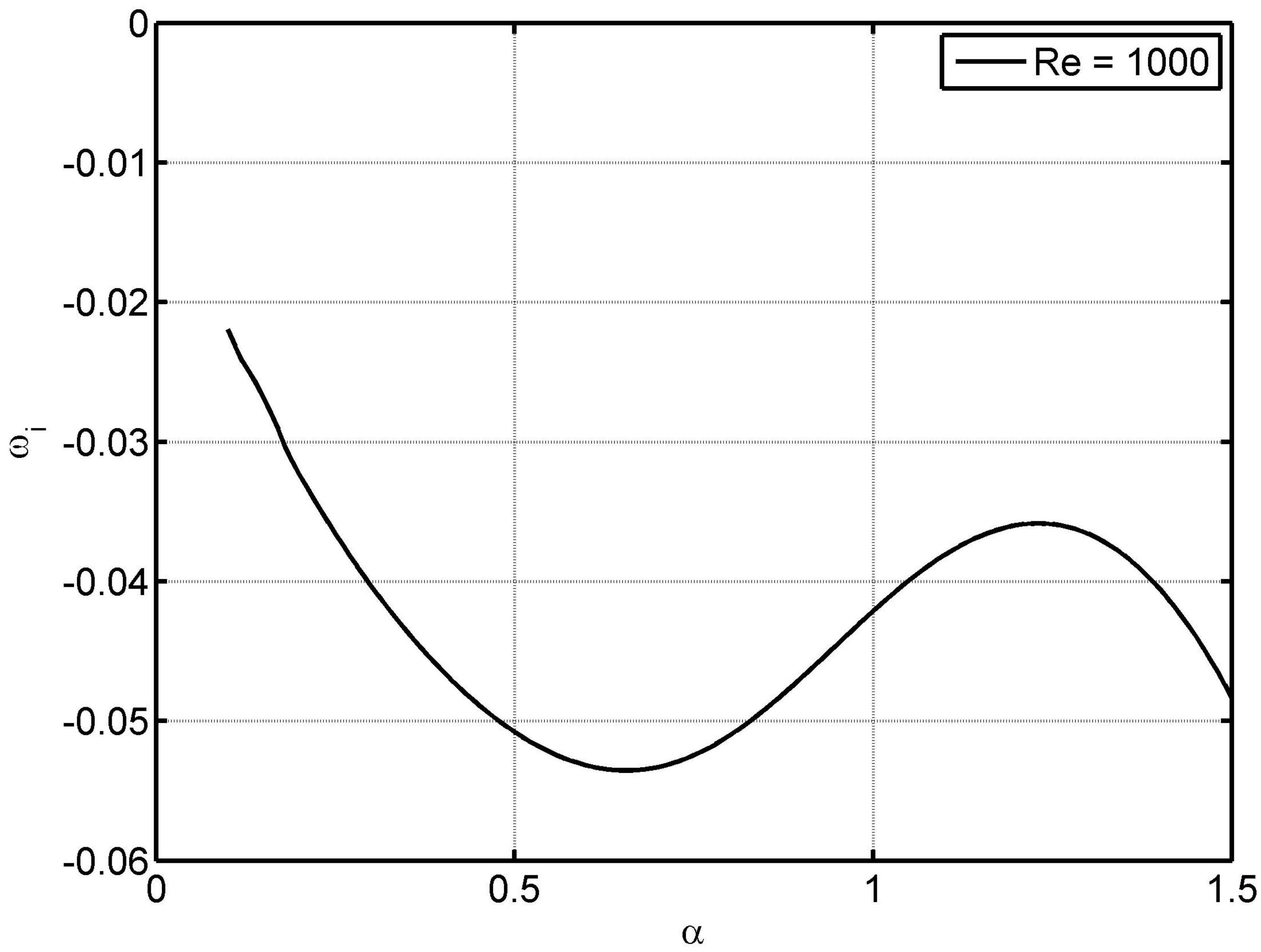
Some errors remain in this region of high sensitivity of the O-S operator

Results - convergence of most unstable mode

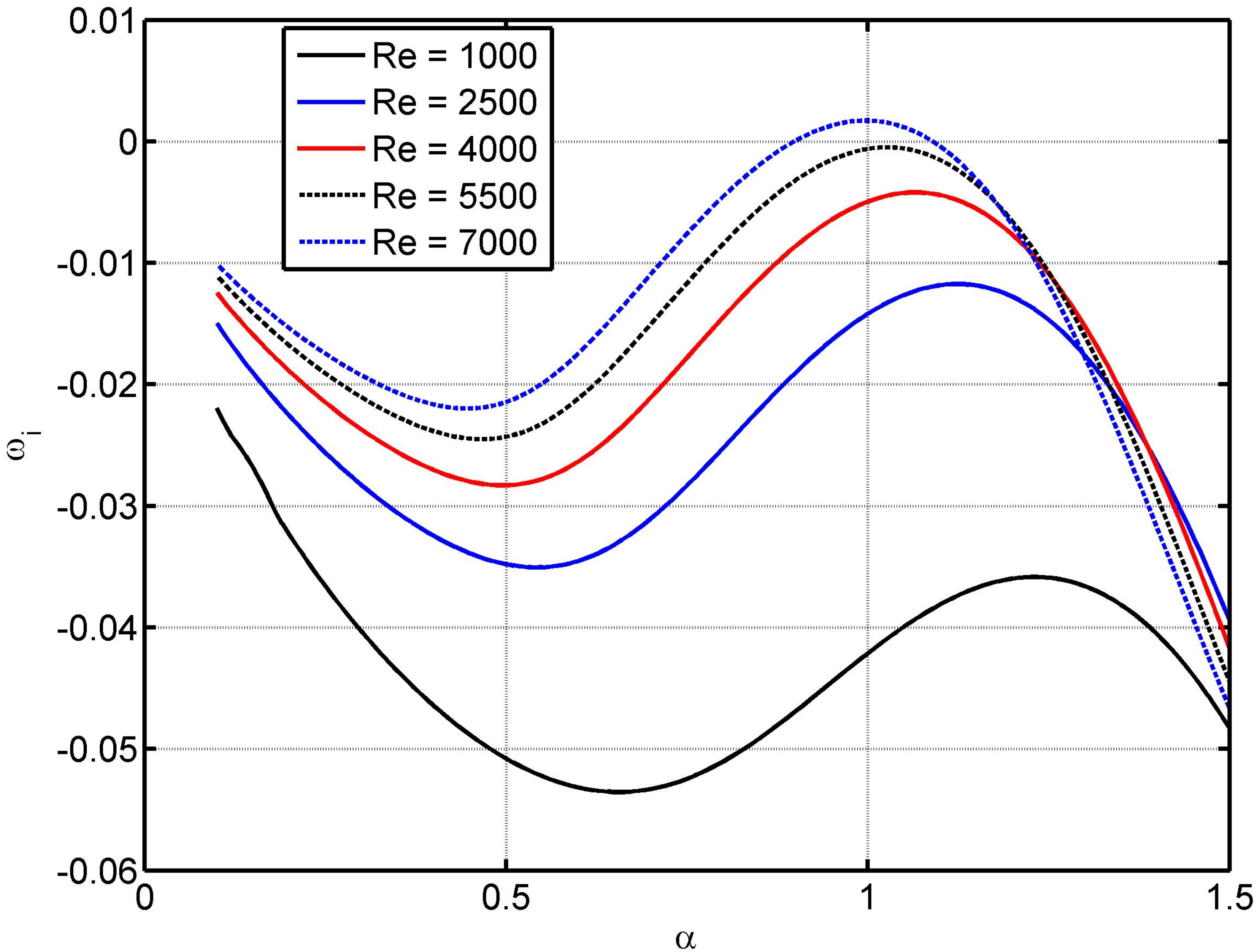


Orszag 1971: $0{\cdot}23752649 + 0{\cdot}00373967i$

Results - growth rate of most unstable mode & effect of Re



Results - growth rate of most unstable mode & effect of Re

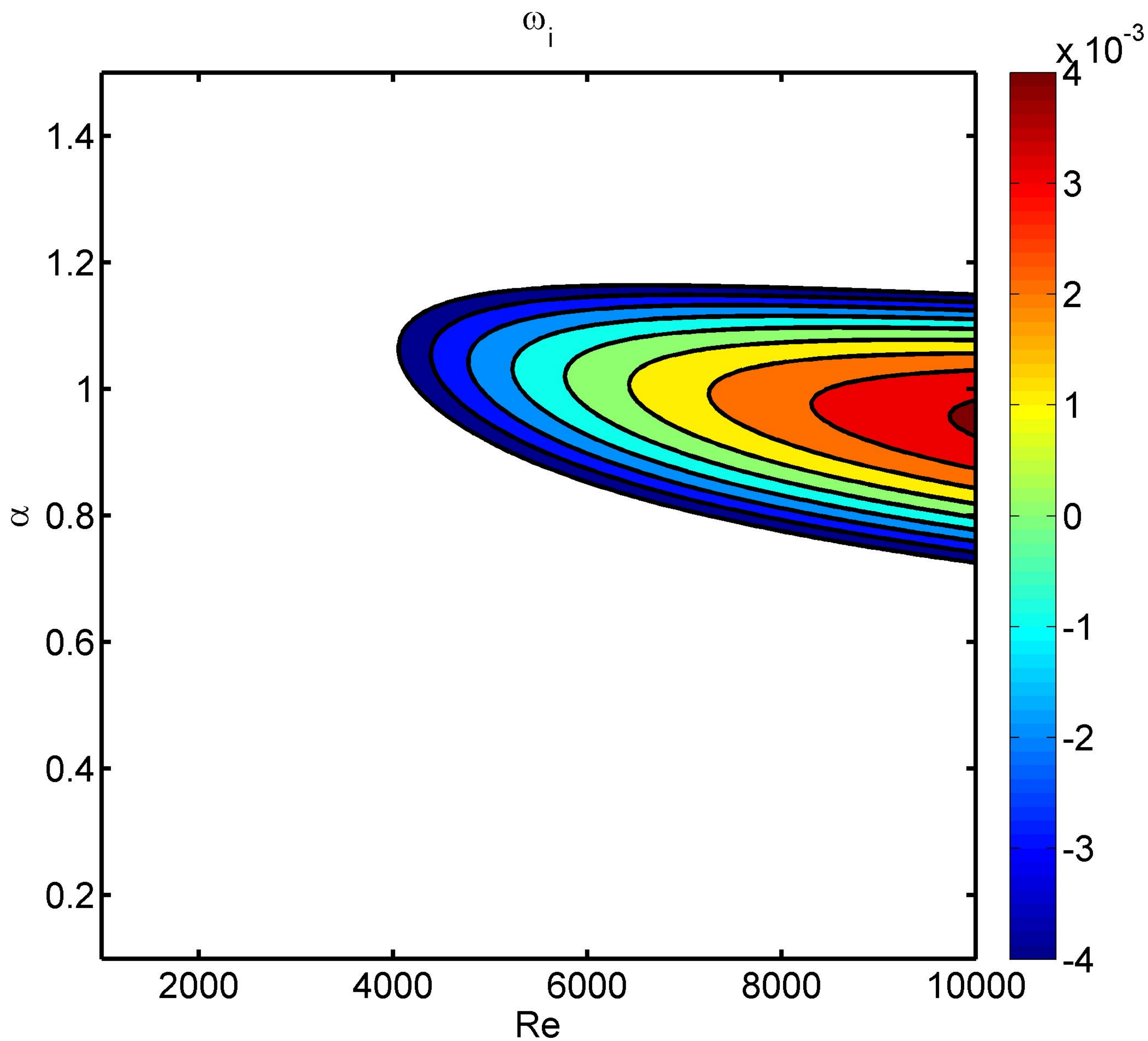


Low Re is stabilising (viscous damping)

Below a certain Re all modes are stable

Above a certain Re one mode becomes unstable - sufficient for transition

Results - growth rate of most unstable mode & effect of Re

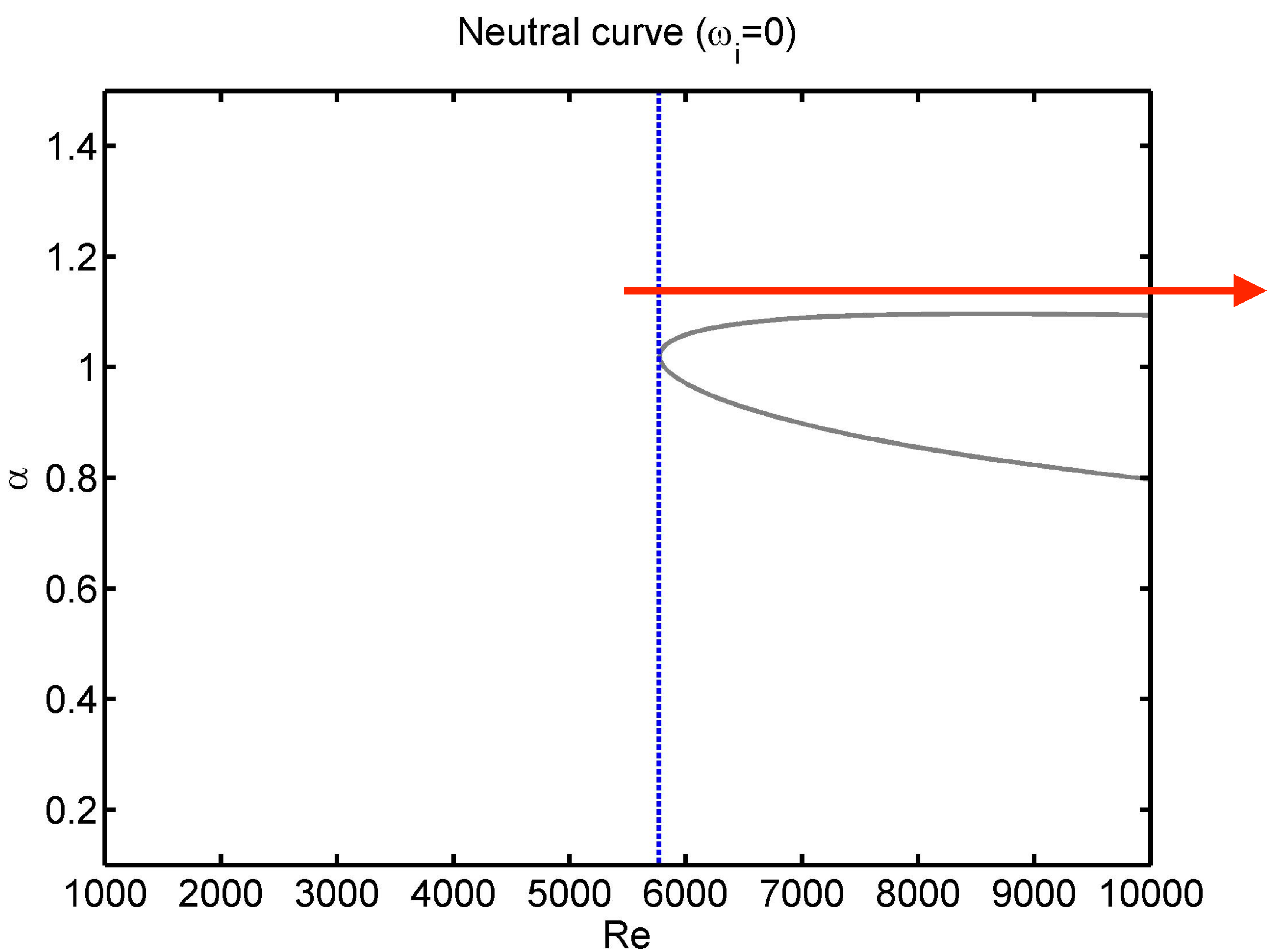


Temporal growth rate of most unstable mode of Poiseuille flow

Green, yellow, red: unstable

Cyan, blue, white: stable

Results - growth rate of most unstable mode & effect of Re



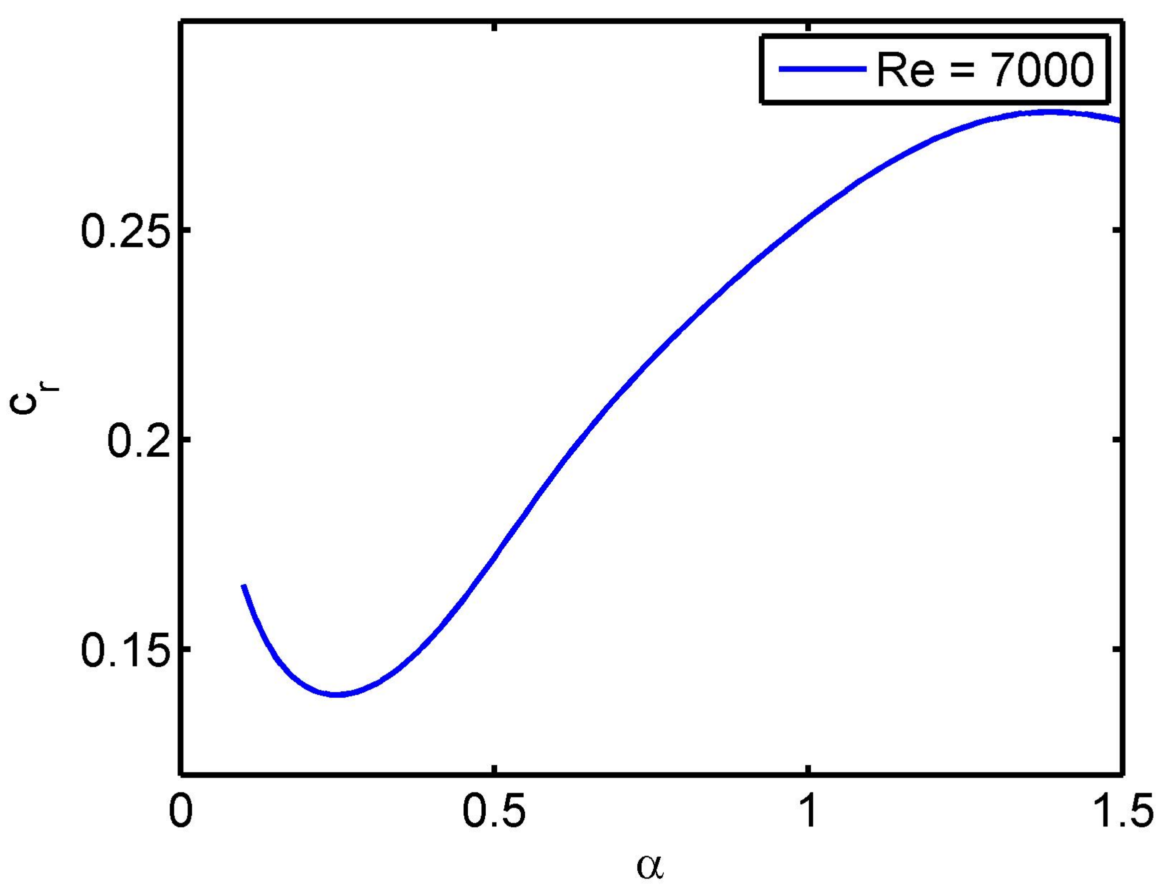
Critical Reynolds number

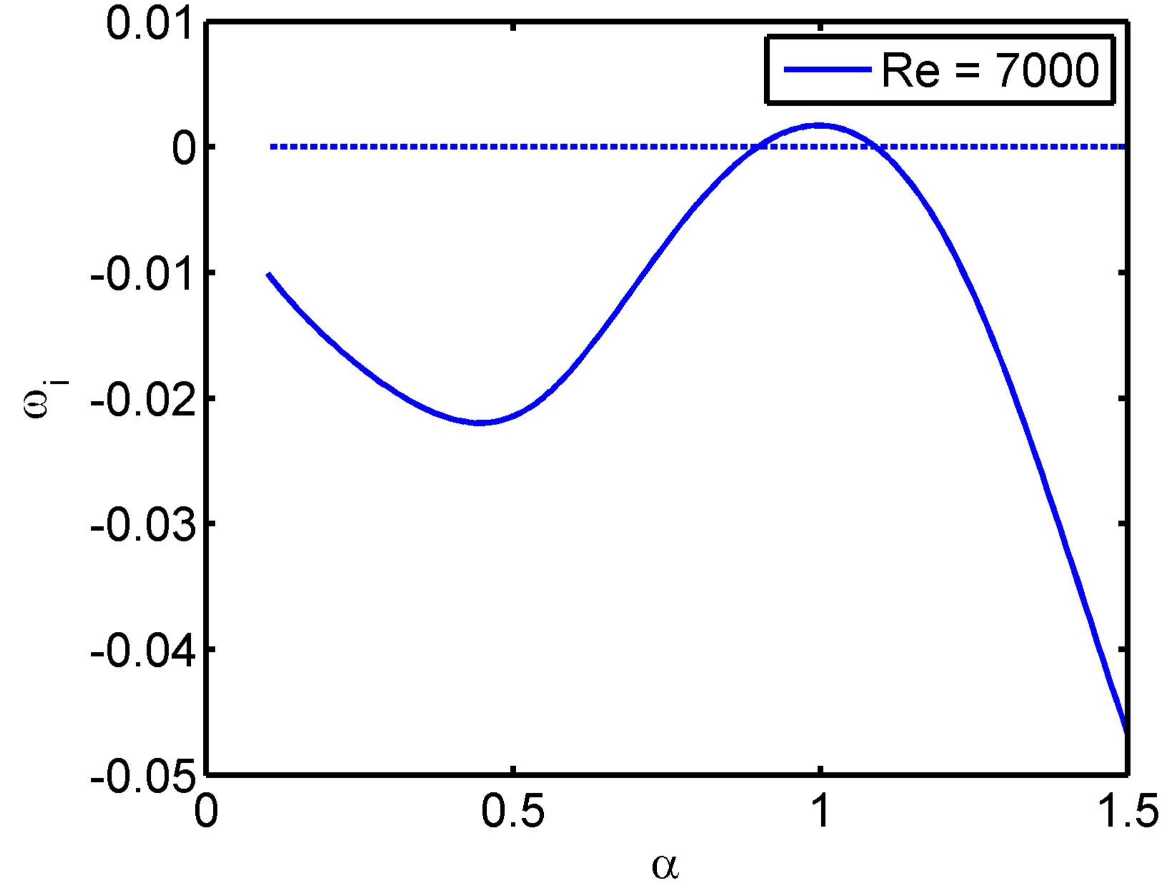
Above this value a ·ange of wavenumbers vill grow exponentially with time

Initial stage of ransition to turbulence

Results - phase speed, showing dispersive nature of instability waves (unlike those of Rayleigh equation)

Most unstable wave in Poiseuille flow





Each mode has a dispersion relation $\omega = \omega(\alpha)$

Experimental results

Nishioka, Iida & Ichikawa 1975 – channel flow, background turbulence=0.05%

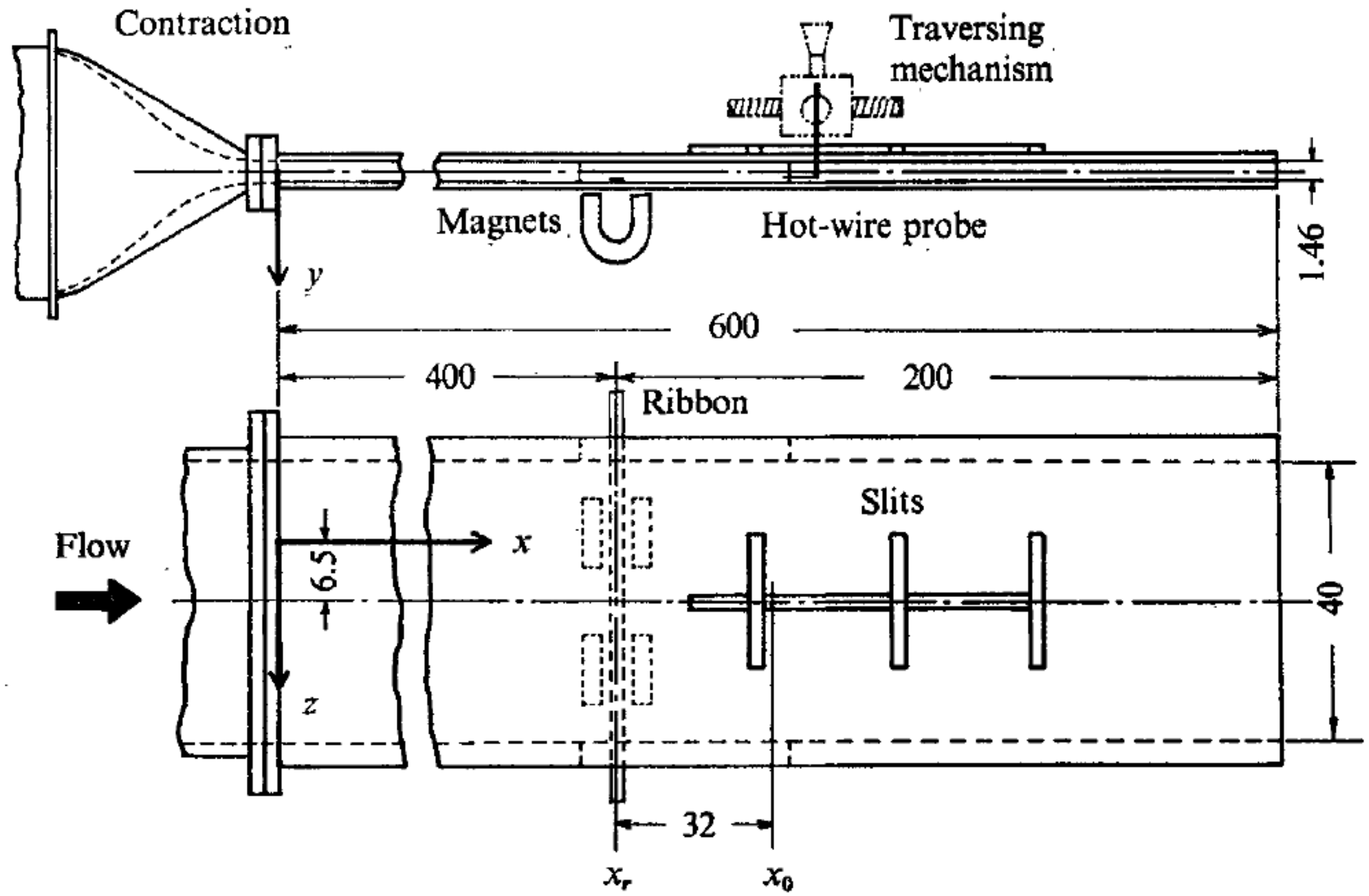


FIGURE 1. Channel apparatus and co-ordinate system. Dimensions in centimetres.

Experimental results

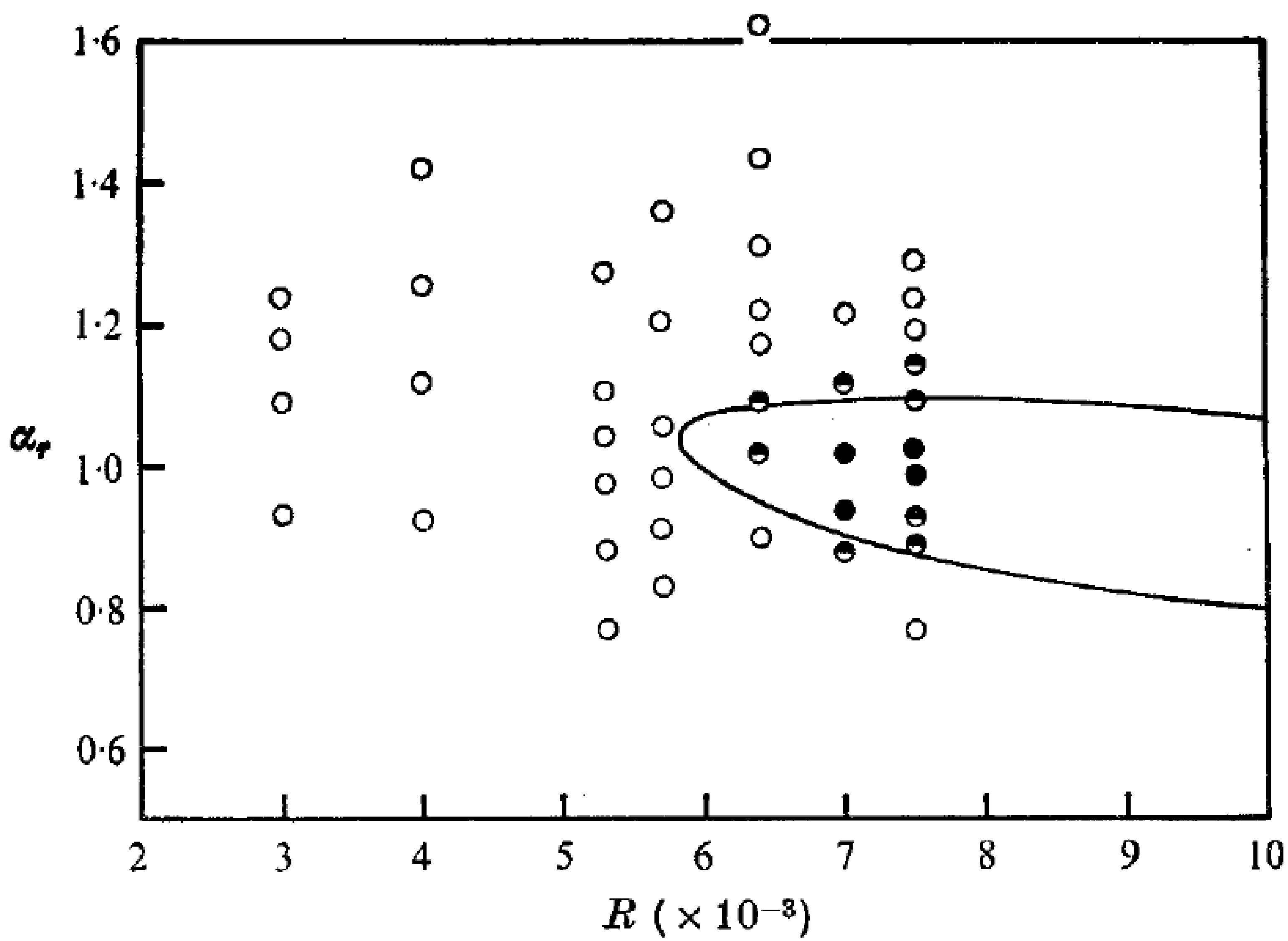


FIGURE 11. Stability boundary for small disturbances. ——, Ito's neutral curve. Our results: ○, damped; ◓, nearly neutral; ●, amplified.

Note: other experiments report transition ag lower Re

Patel & Head 1969: at Re=2500

Karnitz, Potter & Smith 1974: at Re=5000 (background turbulence: 0.3%)

Subcritical transtion: more on that later

Exercises:

- Evaluate the Reynolds-number effect on the temporal growth rate for a mixing layer (tanh profile, Michalke 1964). Is it possible to determine a critical Reynolds number?

- Study the temporal stability of the Blasius boundary layer.

- Study the effect of favourable and adverse pressure gradients in the stability boundary layers using the Falkner-Skan family of velocity profiles.

3. Orr-Sommerfeld for the mixing layer

Orr-Sommerfeld equation

$$(U-c)\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right]\hat{v}-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\hat{v}=\frac{1}{i\alpha\mathrm{Re}}\left[\frac{\mathrm{d}^4}{\mathrm{d}y^4}-2\alpha^2\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\alpha^4\right]\hat{v}$$

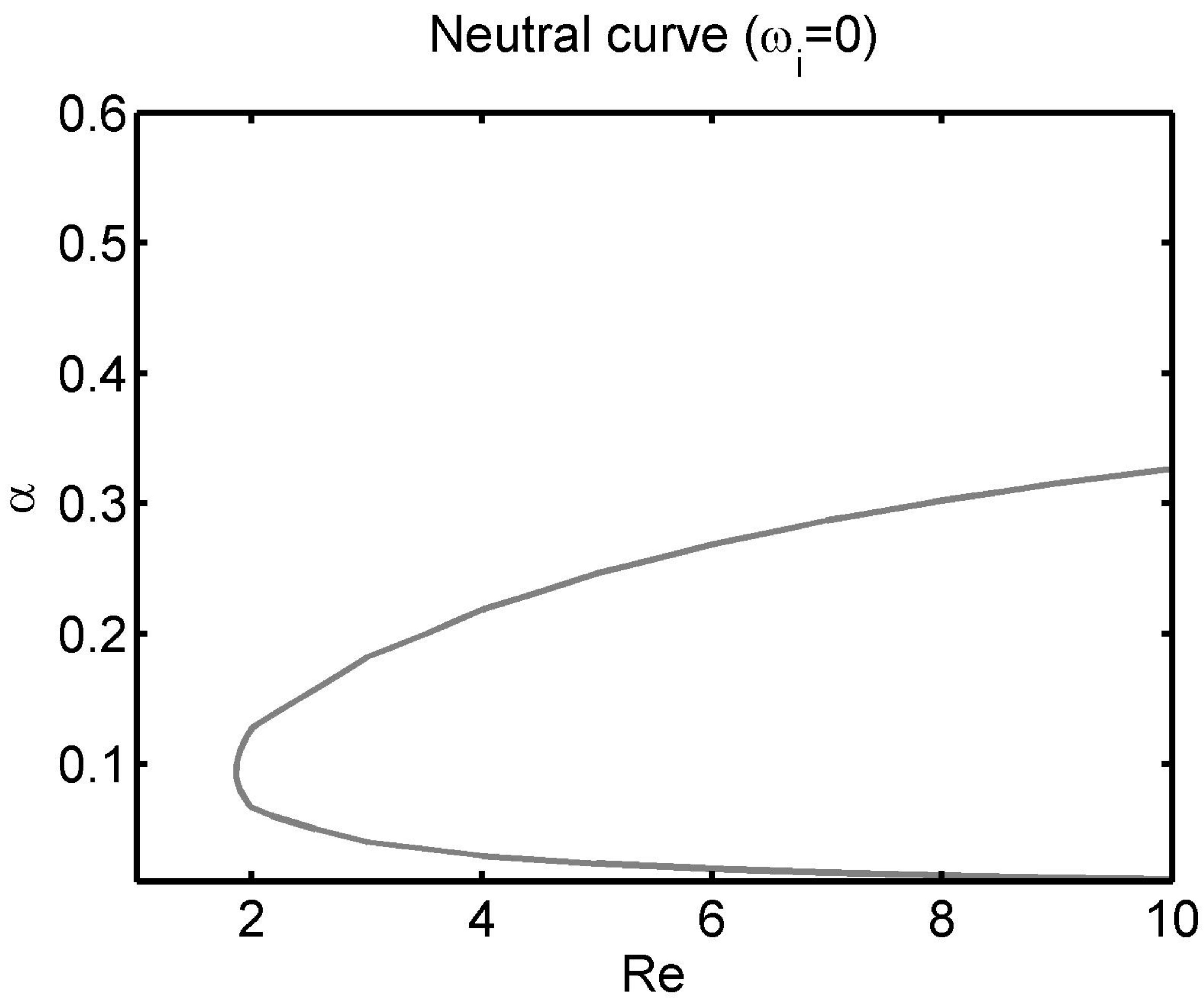
Rearrange to take form of eigenvalue problem

$$\left[U\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right)-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}-\frac{1}{i\alpha\mathrm{Re}}\left(\frac{\mathrm{d}^4}{\mathrm{d}y^4}-2\alpha^2\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\alpha^4\right)\right]\hat{v}=c\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right]\hat{v}$$

$$\mathbf{L}\mathbf{v}=c\mathbf{F}\mathbf{v}$$

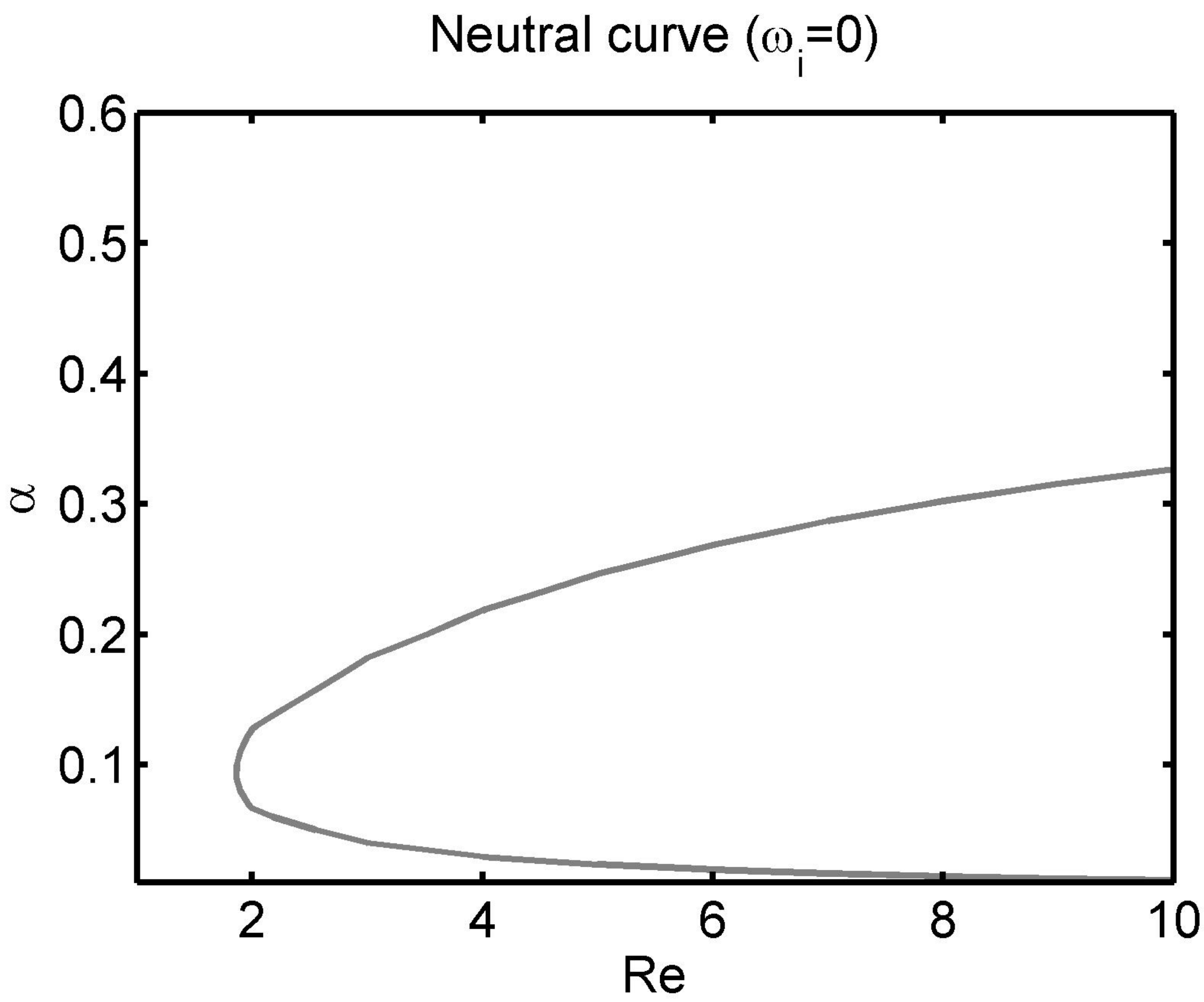
Exercise: write a code to solve the temporal stability problem for the mixing layer (U=0.5(1+tanh(y))), with BCs: v=dv/dy->0 for y->infinity. Is it possible to find critical Reynolds number?

Results



Does this make sense?

Results



Does this make sense?

Boundary-layer theory:

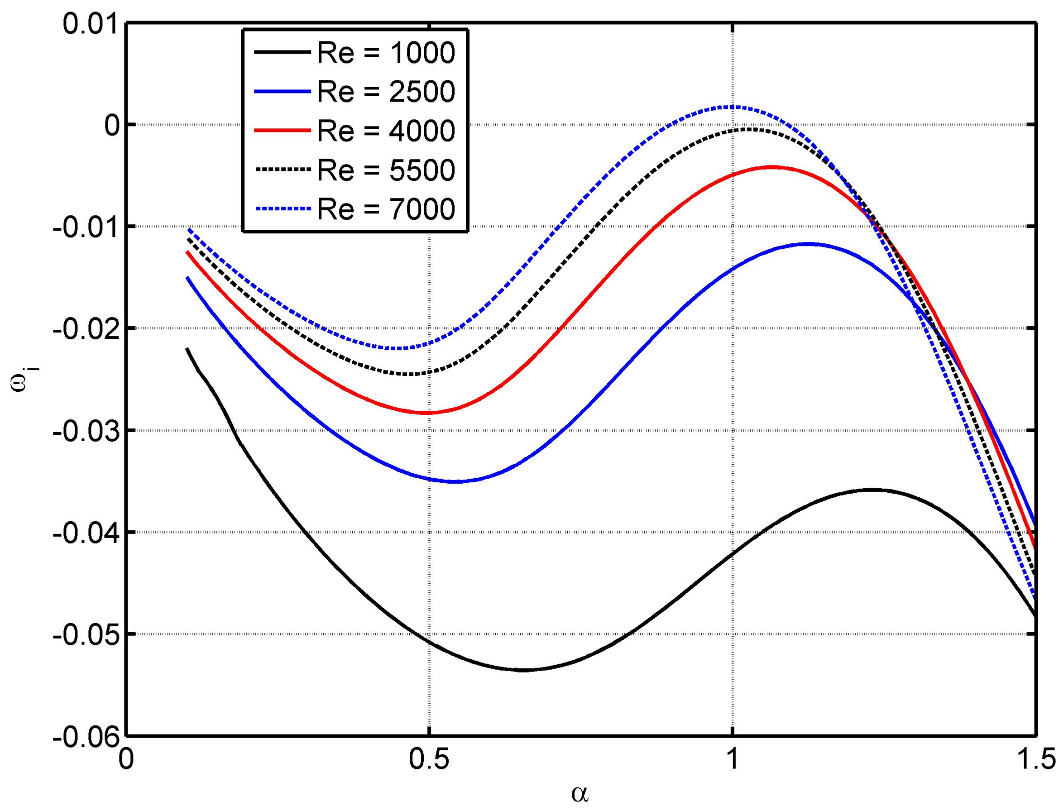
$$\delta \approx \frac{x}{\sqrt{\mathrm{Re}}}$$

But we assume $U=U(y)$

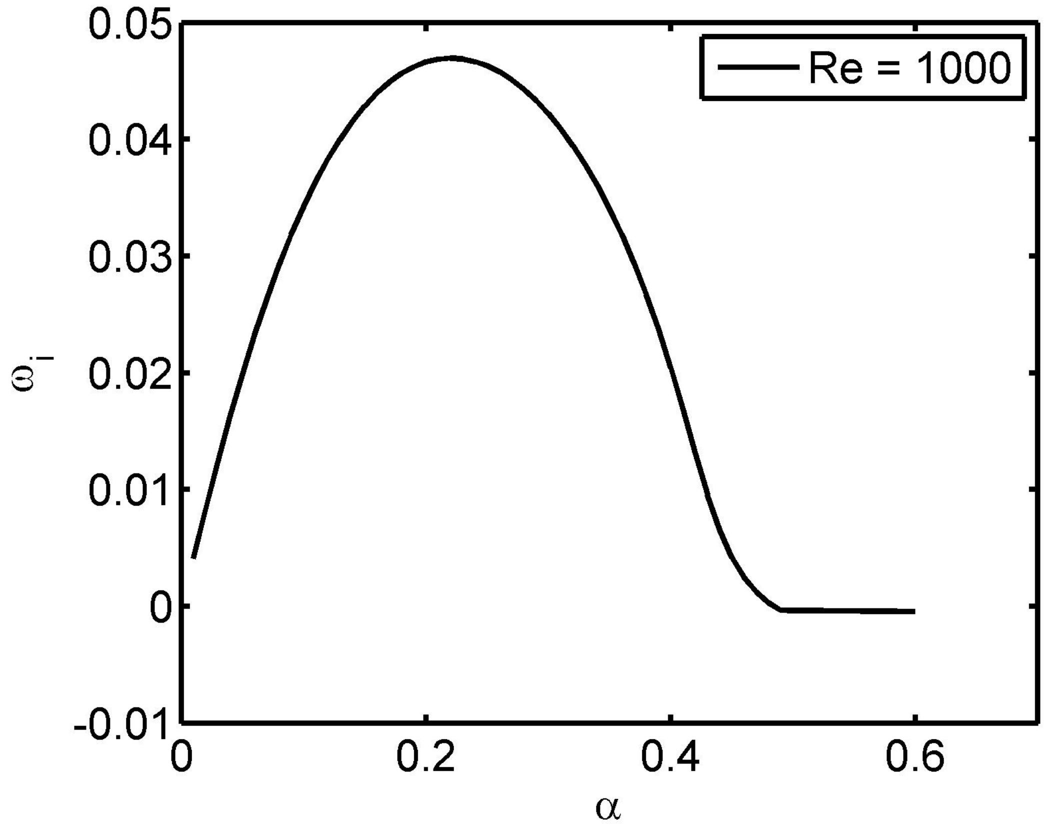
Parallel flow hypothesis only applicable for Re>>1

Compare plane Poisueille and mixing-layer flows

Plane Poisueille flow



Mixing layer flow



- High crit. Re
- Small range of unstable wavenumbers
- Stable for Re-> infinity

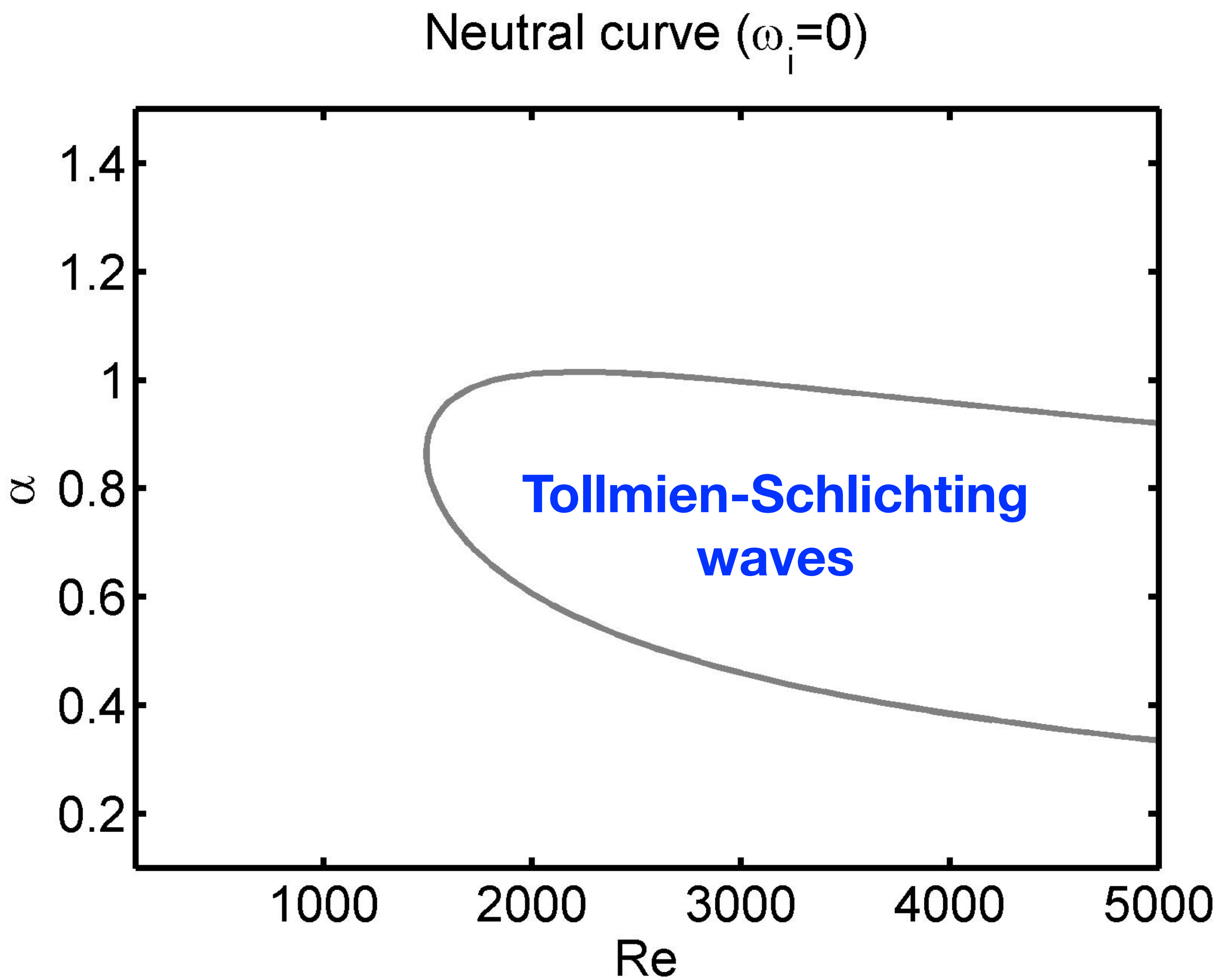
- No crit. Re
- Large range of unstable wavenumbers
- Unstable for Re-> infinity

WHY?

4. Instability in boundary layers

- Blasius
- Falkner-Skan

Blasius boundary layer $\beta = 0$



Tollmien 1931, 1936
Schlichting 1932, 1933, 1935

Blasius boundary layer $\beta = 0$

Visualisation of T-S waves

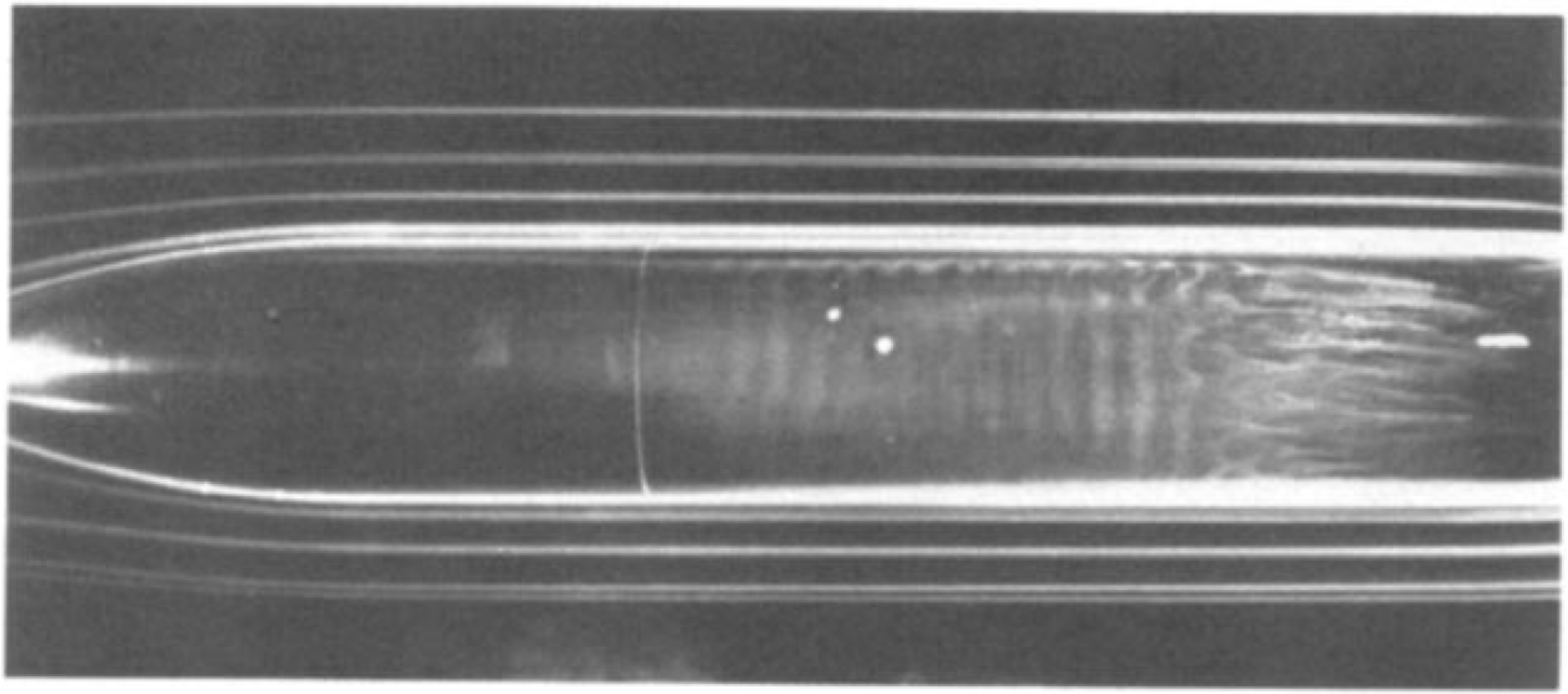


Figure 1 Smoke-flow visualization in the boundary layer over an axisymmetric body. Photograph by F. N. M. Brown (courtesy of the University of Notre Dame).

Experimental observations

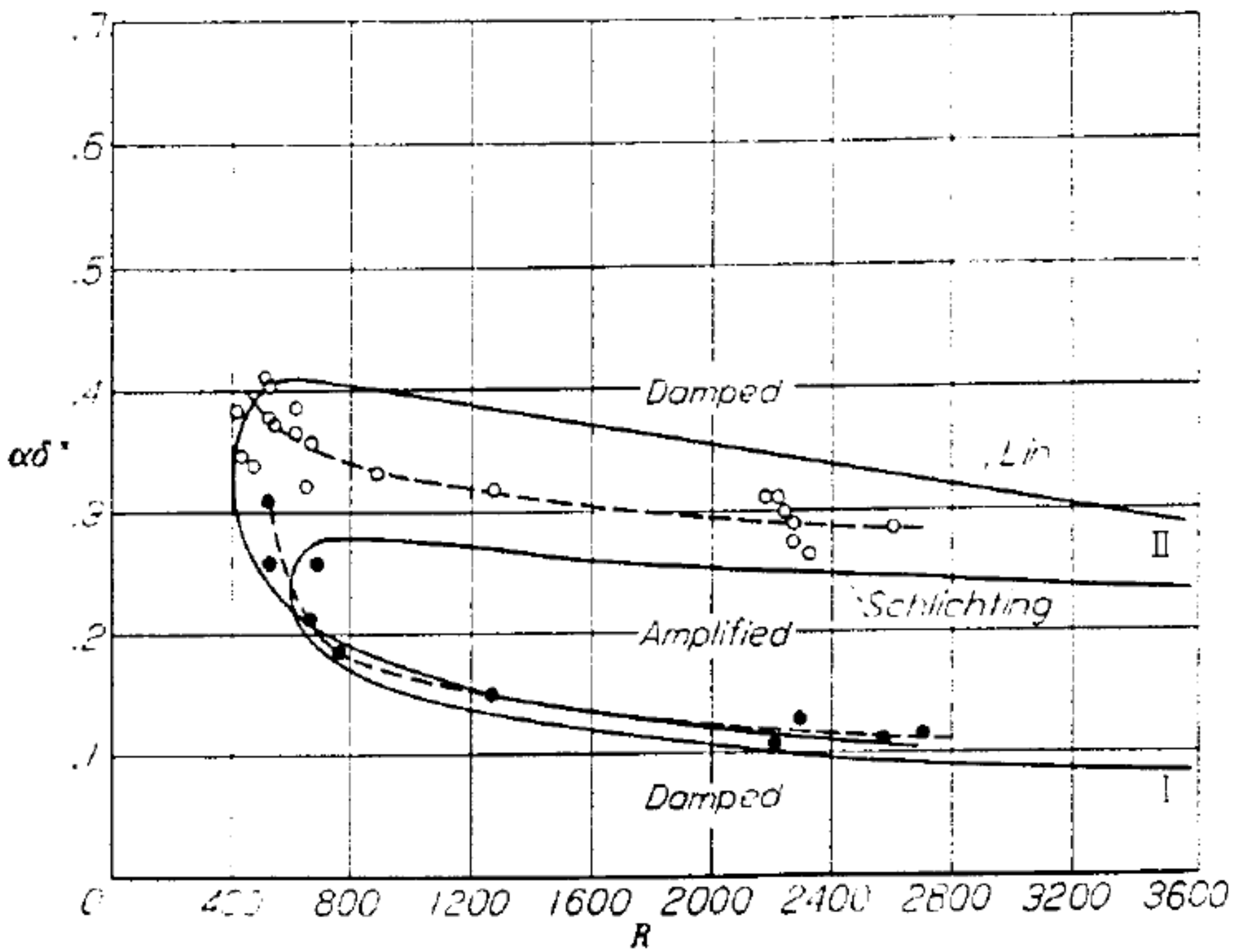


FIGURE 25.—Wave length of neutral oscillations excited in boundary layer by vibrating ribbon. Theoretical neutral curves shown solid, neutral curves defined by experimental points shown broken. Closed circles, branch I. Open circles, branch II.

Note: Reynolds number based on displacement thickness

Experimental observations

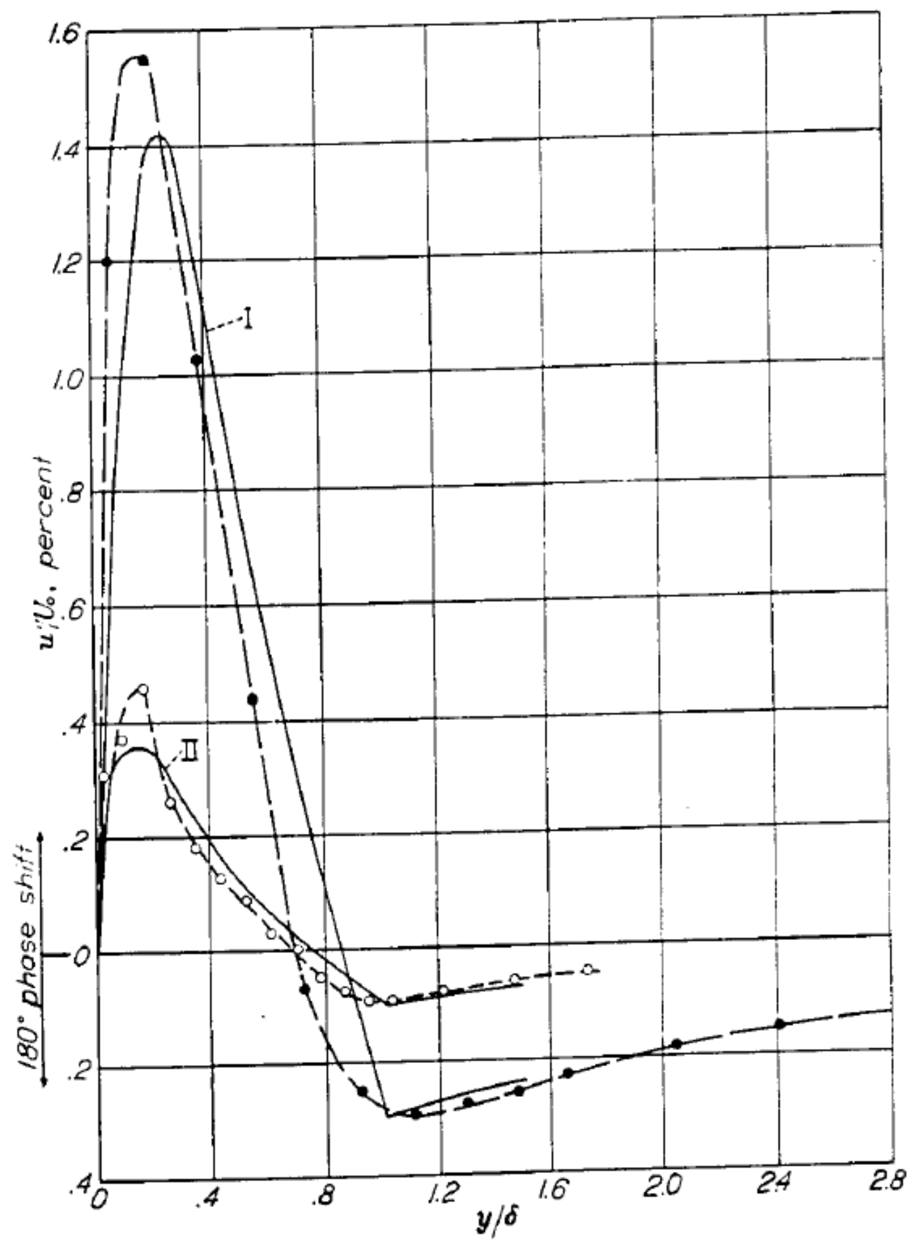


FIGURE 21.—Distribution of amplitude of oscillations across boundary layer. Solid curves are theoretical according to Schlichting. Broken curves are experimental.

Eigenfunctions

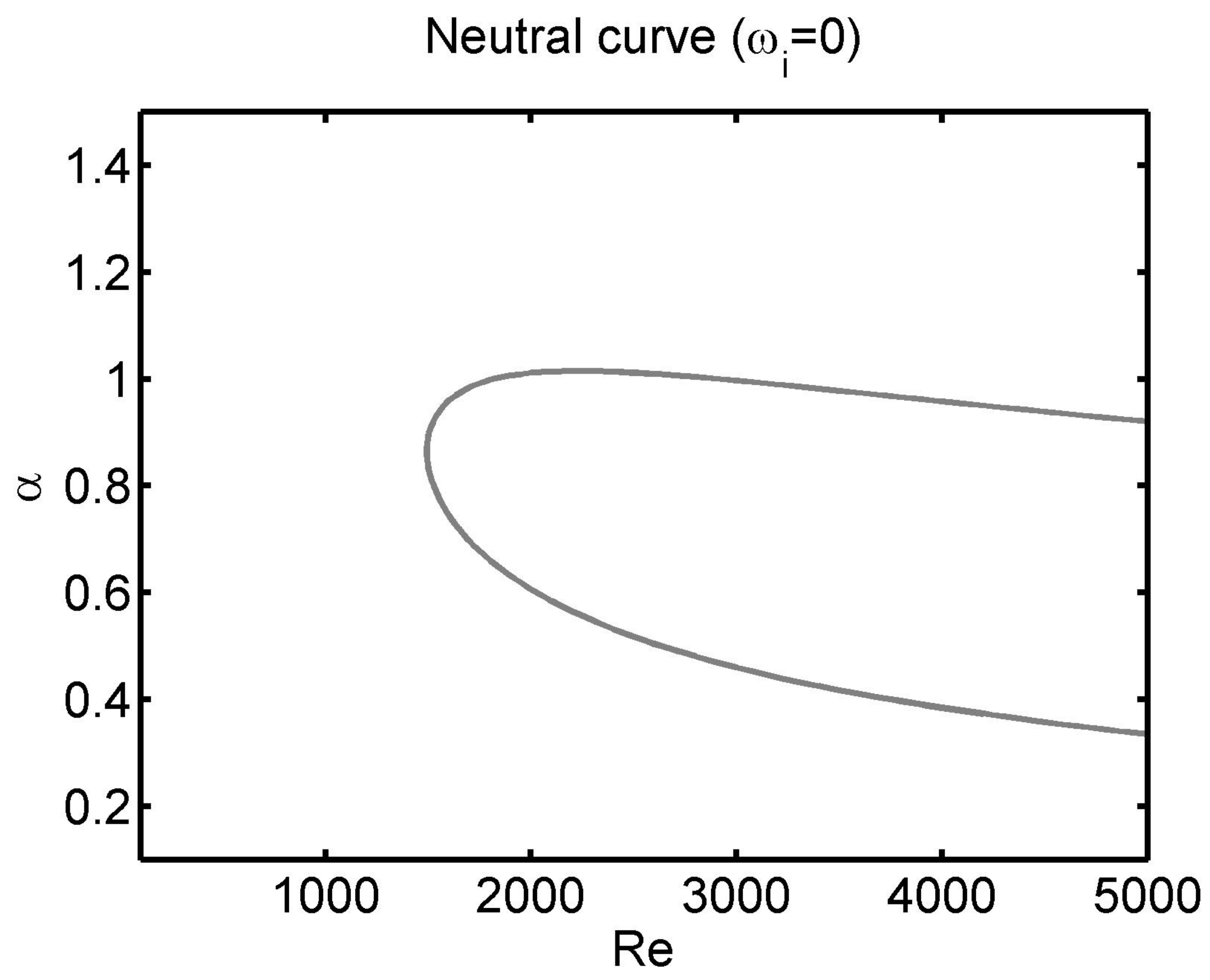
Full lines: experiment
Dashed lines: theory (Schlichting)

All done with only vibrating ribbons and hot wires…

Results

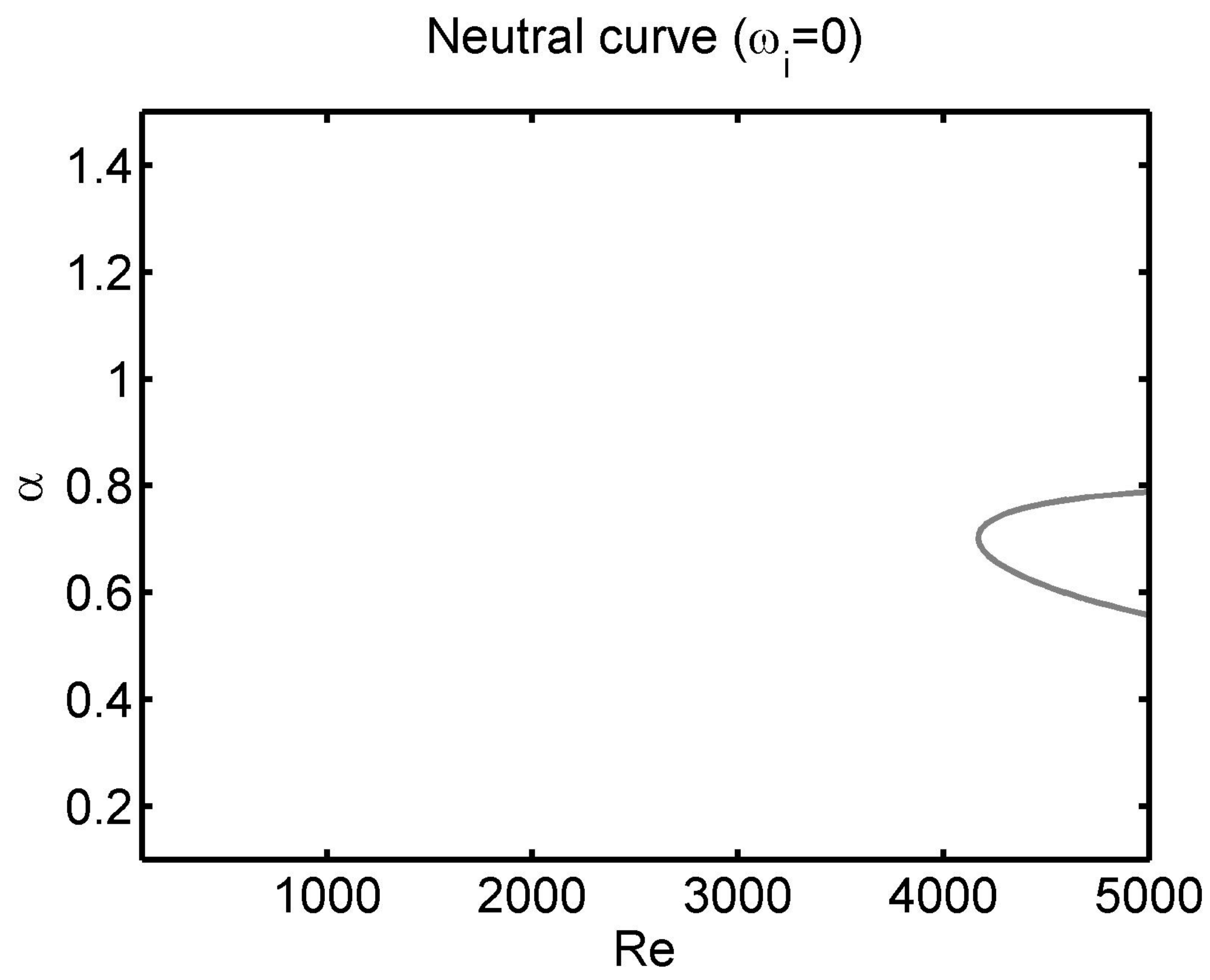
Blasius boundary layer

$$\beta = 0$$



Favourable pressure gradient

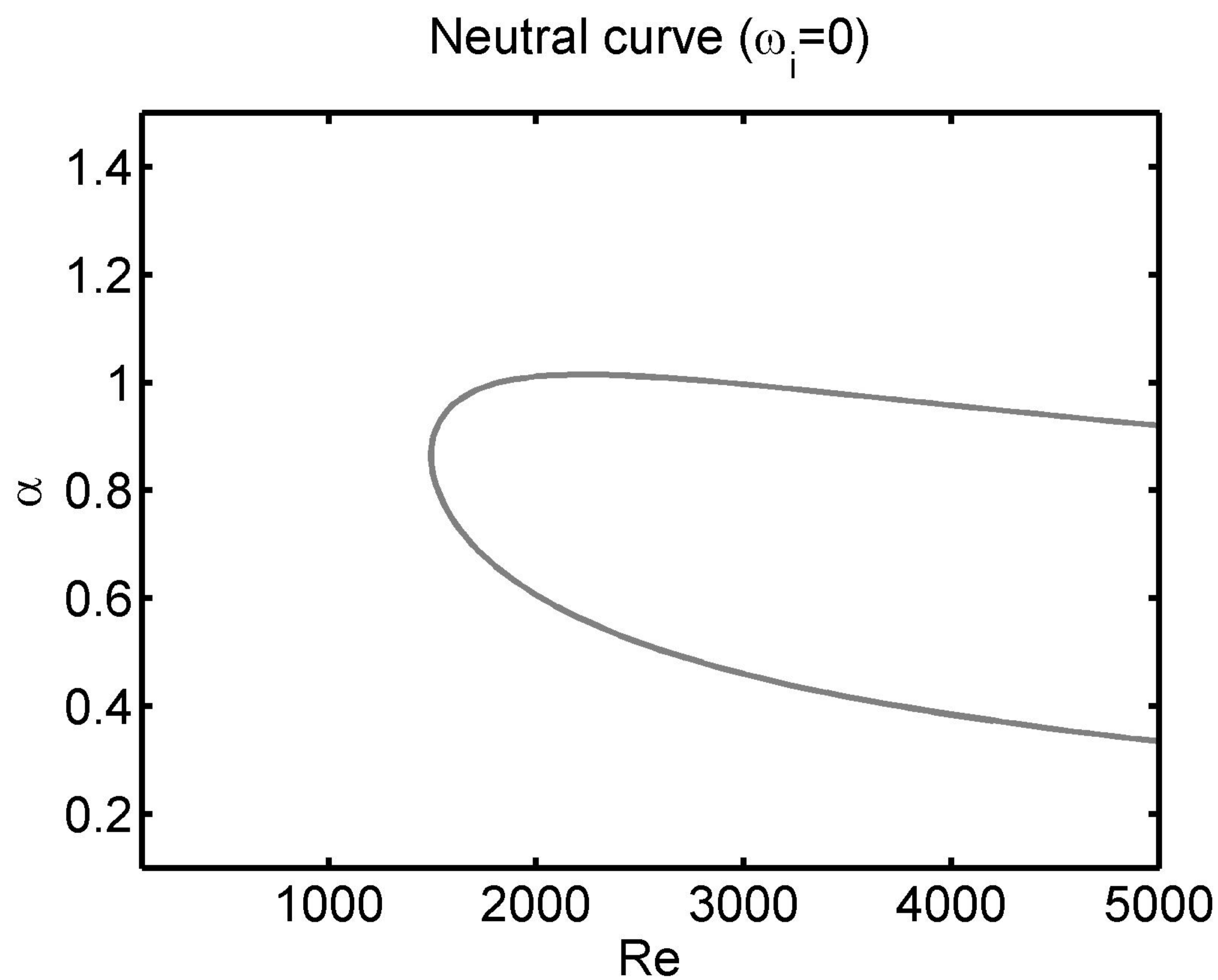
$$\beta = 0.1$$



Results

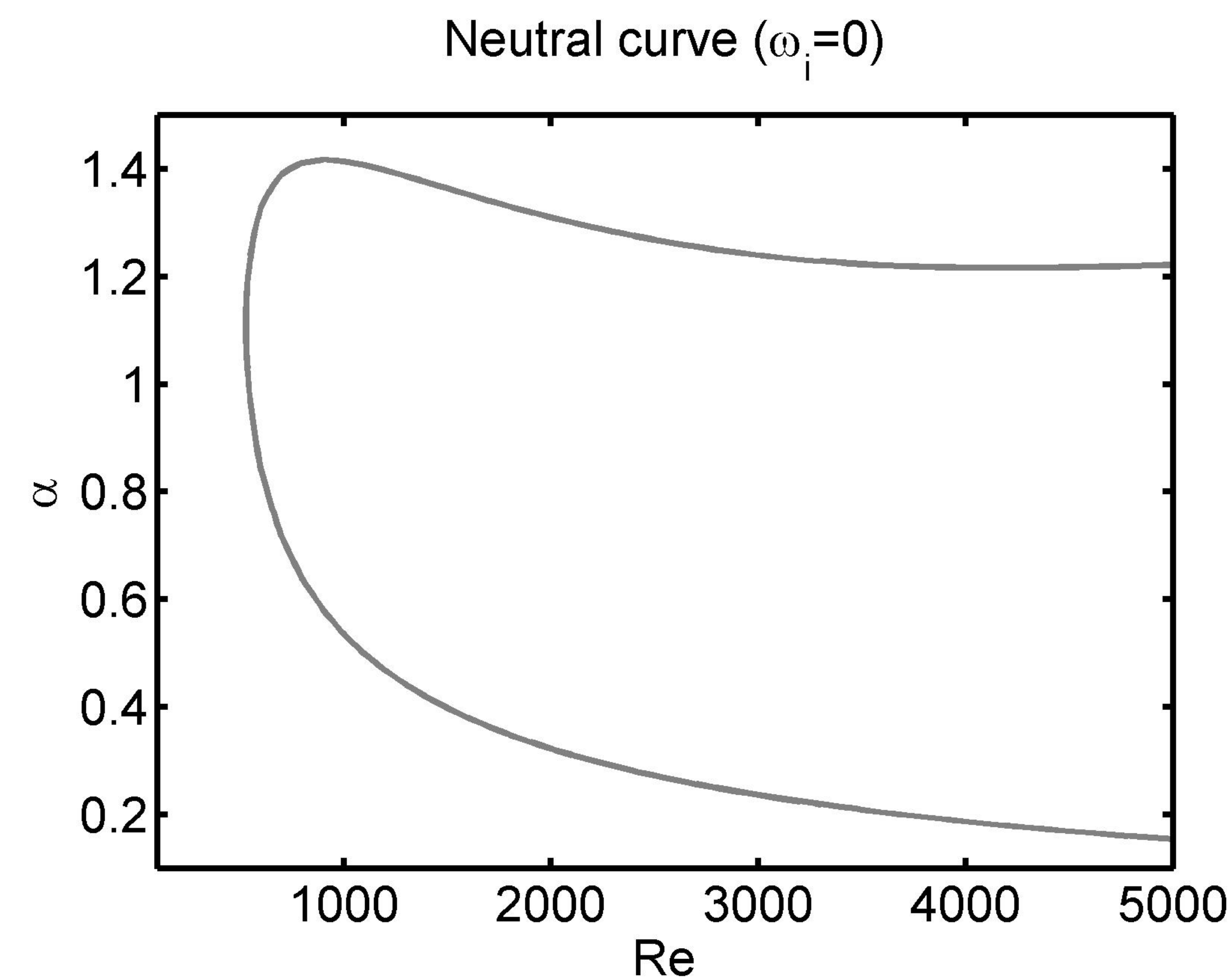
Blasius boundary layer

$$\beta = 0$$



Adverse pressure gradient

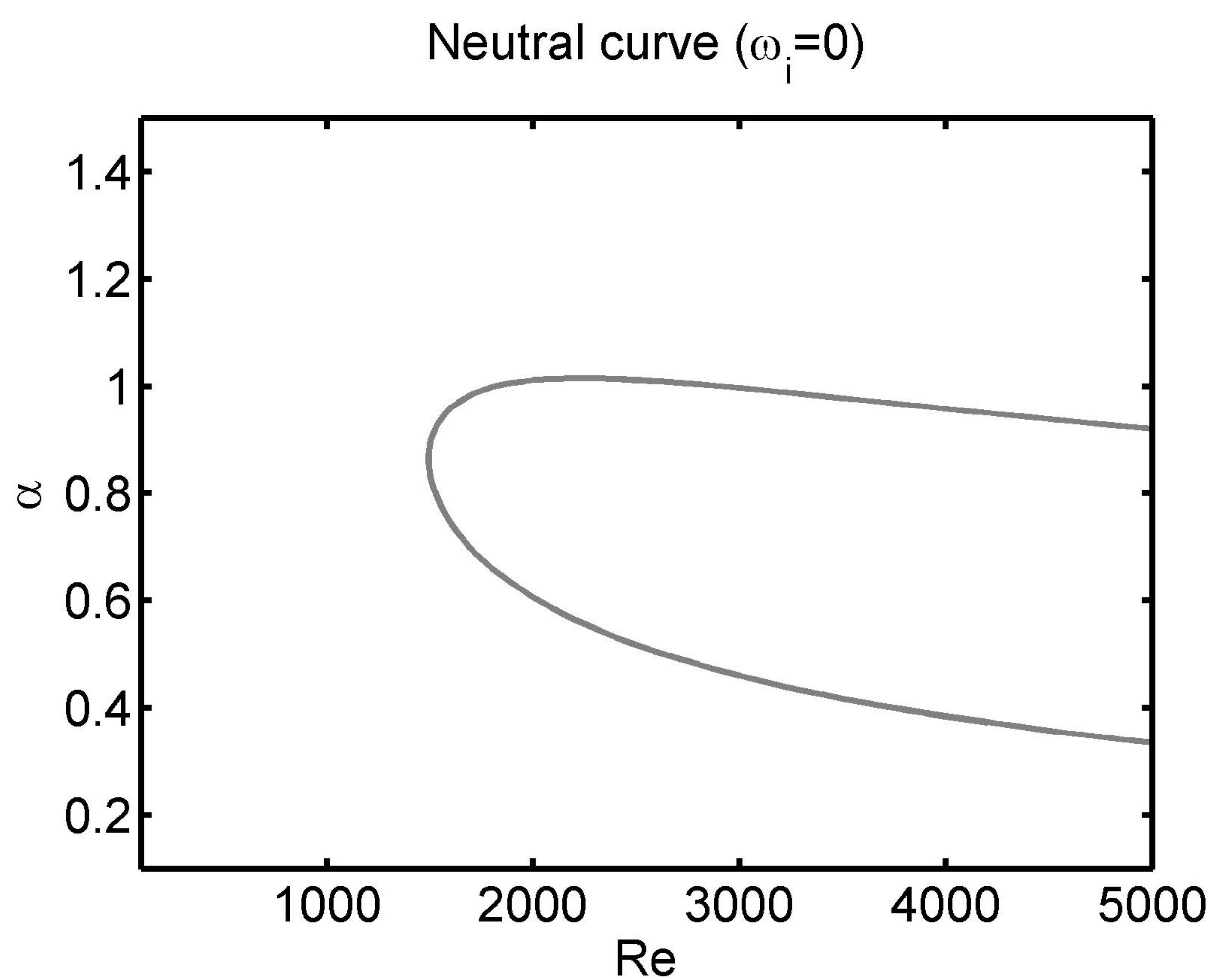
$$\beta = -0.1$$



Results

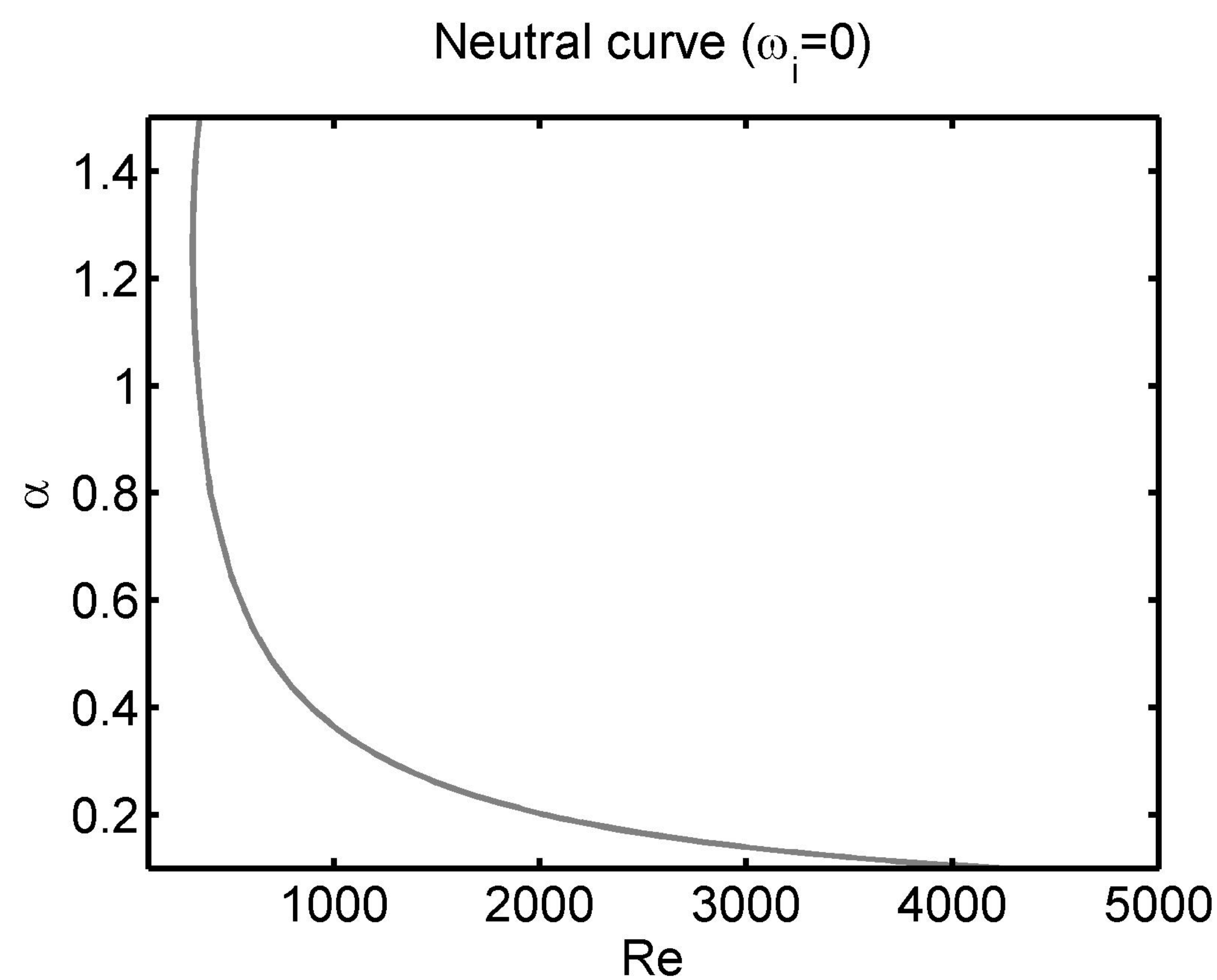
Blasius boundary layer

$$\beta = 0$$



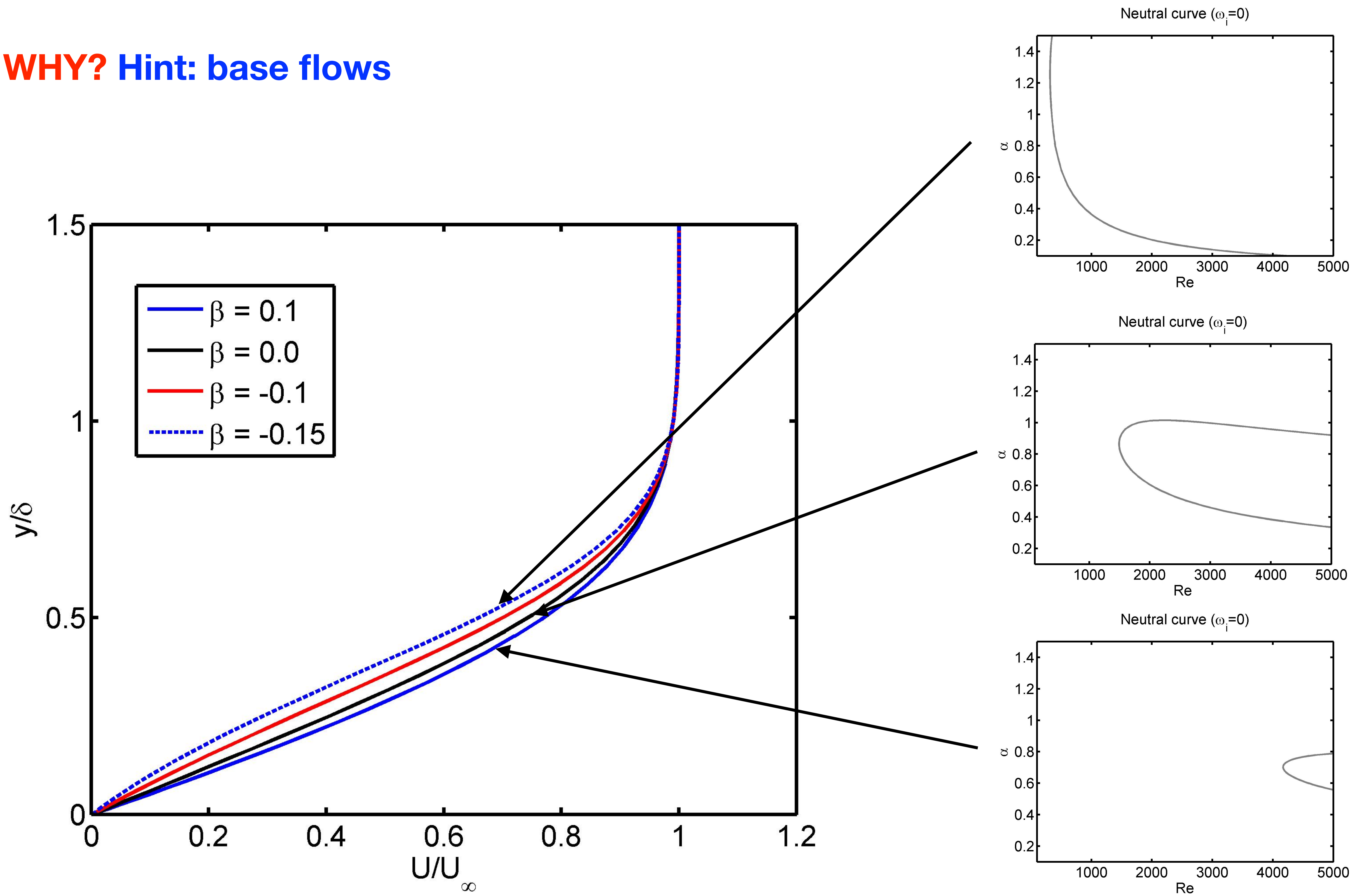
Adverse pressure gradient

$$\beta = -0.15$$

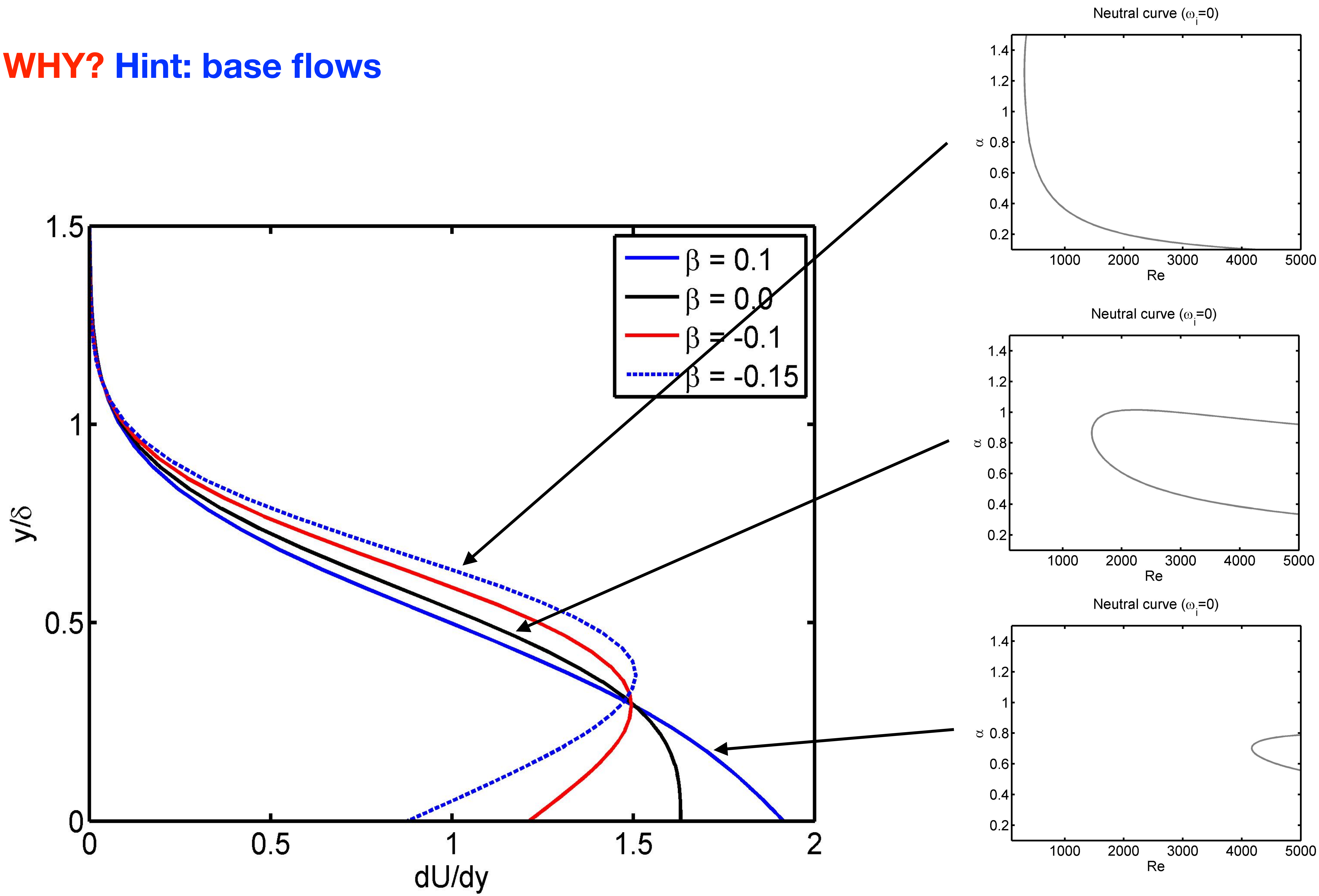


WHY?

WHY? Hint: base flows

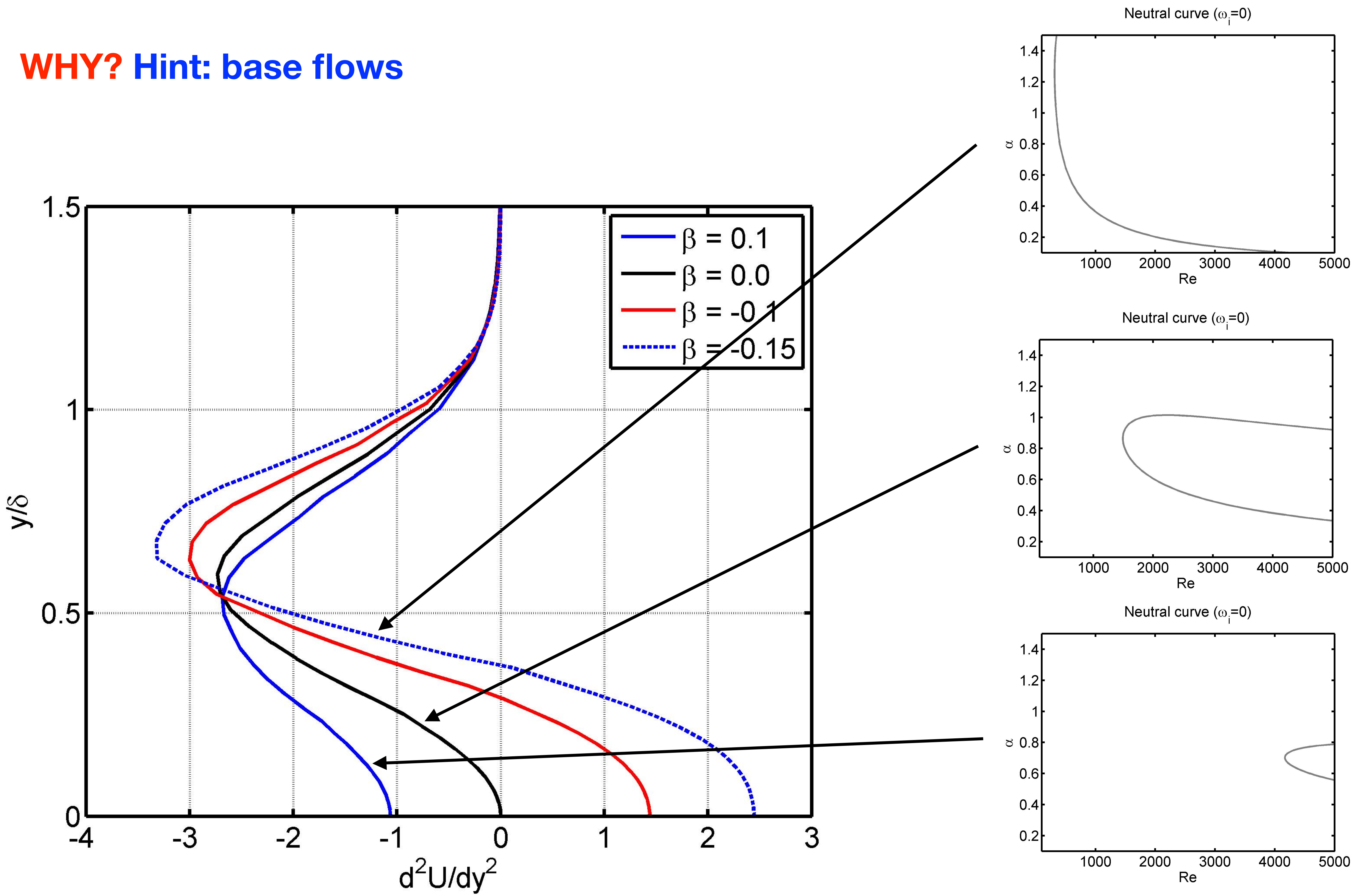


WHY? Hint: base flows



4. Instability in boundary layers

WHY? Hint: base flows



5. Rayleigh's inflection-point theorem

$$(U-c)\left(\frac{\mathrm{d}^2\phi(y)}{\mathrm{d}y^2}-\alpha^2\phi(y)\right)-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\phi(y)=0$$

$$(U-c)(\phi_{yy}-\alpha^2\phi)-U_{yy}\phi=0$$

$$(U - c)(\phi_{yy} - \alpha^2 \phi) - U_{yy}\phi = 0$$

Assume flow is temporally unstable: $c_i > 0$

Integrate equation subject of the boundary conditions:

$$\phi = 0 \qquad \text{at} \qquad y = a, b$$

Bearing in mind that solution implies instability.

Multiply through by complex conjugate of ϕ: ϕ^* and integrate:

$$\int_a^b \left[\phi^*\phi_{yy} - \alpha^2\phi\phi^* - \frac{U_{yy}}{U-c}\phi\phi^*\right]\mathrm{d}y = 0$$

Integrate the first term by parts:

0 (boundary conditions)

$$\int_a^b \phi^*\phi_{yy}\mathrm{d}y = \phi^*\phi_y\big|_a^b - \int_a^b \phi_y^*\phi_y\mathrm{d}y$$

$$\int_a^b \left[\phi^*\phi_{yy} - \alpha^2\phi\phi^* - \frac{U_{yy}}{U-c}\phi\phi^*\right]\mathrm{d}y = 0$$

$$\int_a^b [|\phi_y|^2 + \alpha^2|\phi|^2\mathrm{d}y + \int_a^b \frac{U_{yy}}{(U-c)}\phi|^2\mathrm{d}y = 0$$

Multiply the numerator and denominator of the second integral by $(U - c^*)$

5. Rayleigh's inflection-point theorem

$$\int_a^b [|\phi_y|^2 + \alpha^2|\phi|^2 \mathrm{d}y + \int_a^b \frac{U_{yy}}{(U-c)} \phi|^2 \mathrm{d}y = 0$$

$$\int_a^b [|\phi_y|^2 + \alpha^2|\phi|^2 \mathrm{d}y + \int_a^b \frac{U_{yy}(U-c^*)}{|U-c|^2} |\phi|^2 \mathrm{d}y = 0$$

The only complex quantity remaining is c^*

Separate equation into real and imaginary parts.

5. Rayleigh's inflection-point theorem

$$\int_a^b [|\phi_y|^2 + \alpha^2|\phi|^2 \mathrm{d}y + \int_a^b \frac{U_{yy}(U - c^*)}{|U - c|^2}|\phi|^2 \mathrm{d}y = 0$$

$$\int_a^b [|\phi_y|^2 + \alpha^2|\phi|^2 \mathrm{d}y + \int_a^b \frac{U_{yy}(U - c_r)}{|U - c|^2}|\phi|^2 \mathrm{d}y = 0$$

$$c_i \int_a^b \frac{U_{yy}}{|U - c|^2}|\phi|^2 \mathrm{d}y = 0$$

ALL POSITIVE

If the flow is unstable $c_i > 0$ and the equation can only be satisfied if U_{yy} changes sign somewhere on the interval, i.e. there must be an inflection point

$$U_{yy}(y_s) = 0$$

5. Rayleigh's inflection-point theorem

$$\int_a^b [|\phi_y|^2 + \alpha^2|\phi|^2 \mathrm{d}y + \int_a^b \frac{U_{yy}(U-c_r)}{|U-c|^2}|\phi|^2 \mathrm{d}y = 0$$

$$c_i \int_a^b \frac{U_{yy}}{|U-c|^2}|\phi|^2 \mathrm{d}y = 0$$

A necessary (but non sufficient) condition for INVISCID INSTABILITY is that

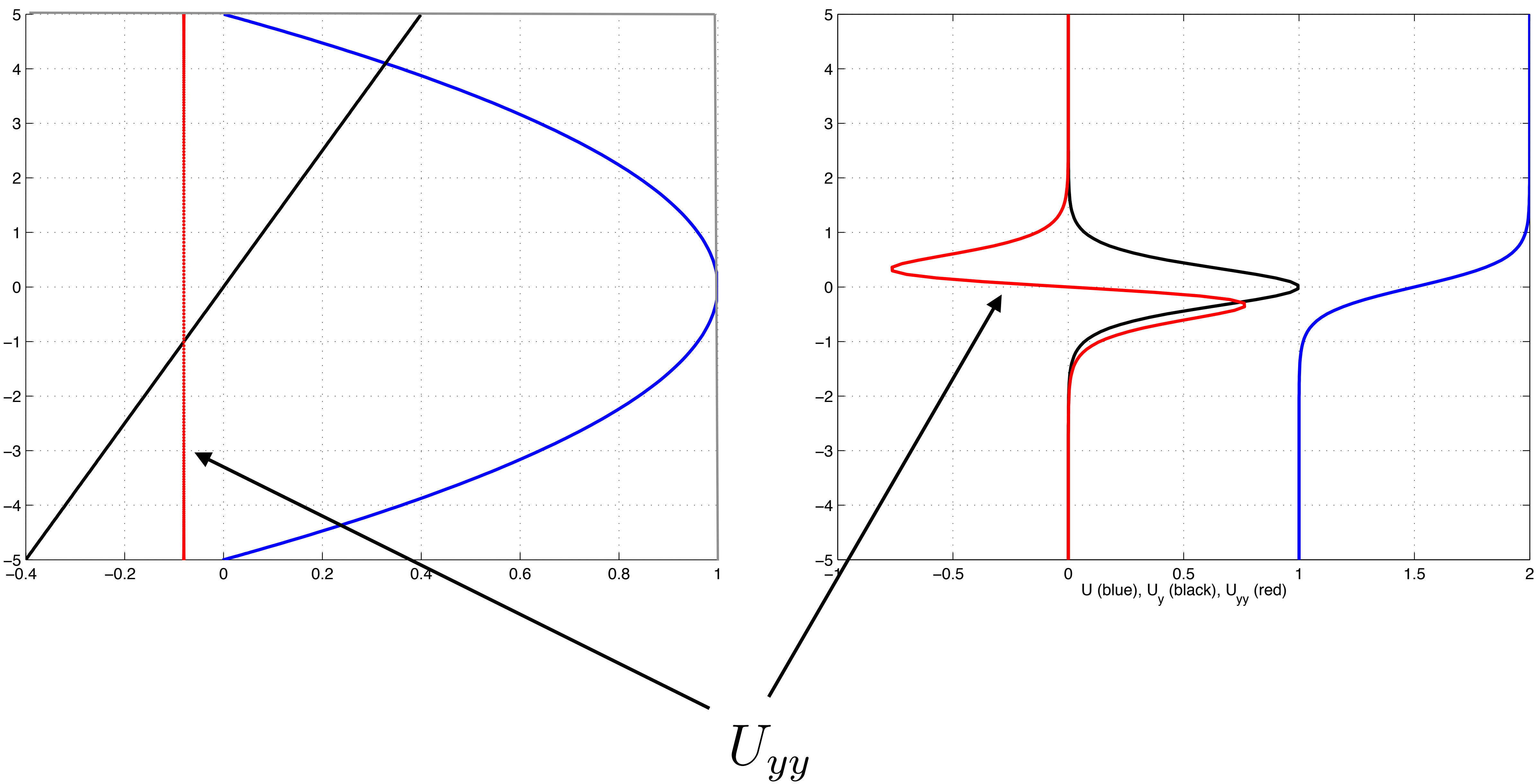
$U_{yy}(y_s) = 0$ **-> Mean curvature (rate of change of vorticity) changes sign.**

A flow without an inflection point will be INVISCIDLY STABLE

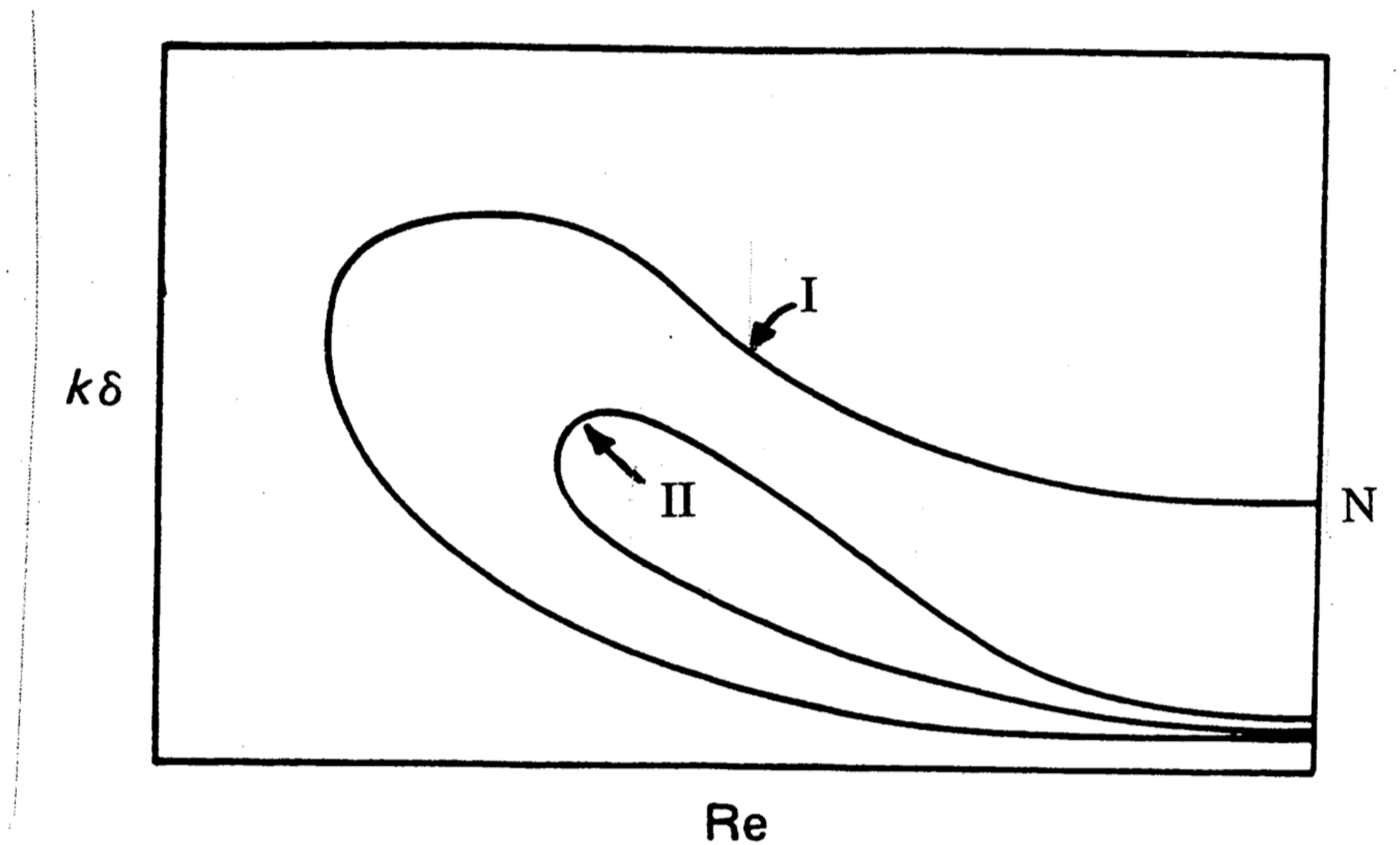
5. Rayleigh's inflection-point theorem

Plane Poisueille flow

Mixing layer flow



Marginal stability curves for inviscidly unstable (I) and inviscidly stable (II) shear-flow profiles



How can viscosity destabilise?

At low Re inertial forces are balanced by viscous forces.

But viscosity acts with a small phase delay.

Use analogy of an oscillator with mass, m, and a linear restoring force proportional to k but with a small time delay, τ

$$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + kx(t-\tau) = 0$$

$$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} - \tau k\frac{\mathrm{d}x(t)}{\mathrm{d}t} + kx(t) = 0$$

Negative damping - destabilising.