I N S T I T U T P P R I M E
CNRS–UPR–3346 • UNIVERSITÉ DE POITIERS • ENSMA

DÉPARTEMENT D2 – FLUIDES
THERMIQUE ET COMBUSTION

An introduction to hydrodynamic stability

Lecture 3: Numerical methods

P. Jordan & A. V. G. Cavalieri*

peter.jordan@univ-poitiers.fr

*Instituto Tecnologico de Aeronautica, Sao, José dos Campos, Brésil

Overview of lecture 3

1. A quick recap. of lecture 2

2. Solving eigenvalue problems

3. Inviscid temporal instability of mixing layer (Kelvin-Helmholtz)

For revision of linear algebra:

https://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/

MATLAB programs:

/MATLAB/

Differentiationmatrices/chebyshev_introduction.m
SimpleEVP/EVPexample_Chebyshev.m
Rayleigh/Rayleigh_tanh_temporal.m
OrrSommerfeld/OS_Poiseuille_eigspec.m

1. Recap. of lecture 2

The general approach for stabilty analysis can be understood by considering a simplified flow configuration:

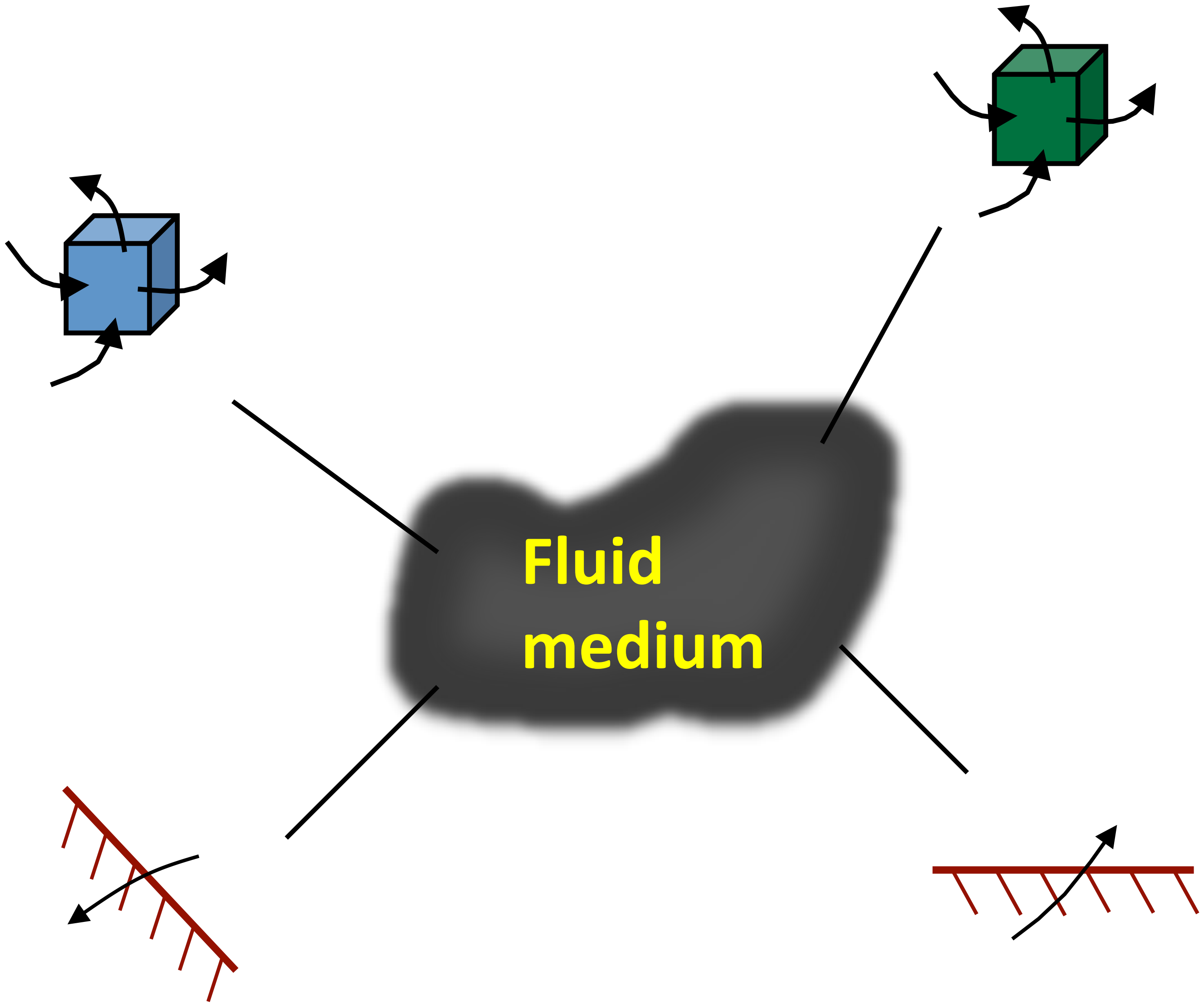
- Parallel, 2D, shear flow,

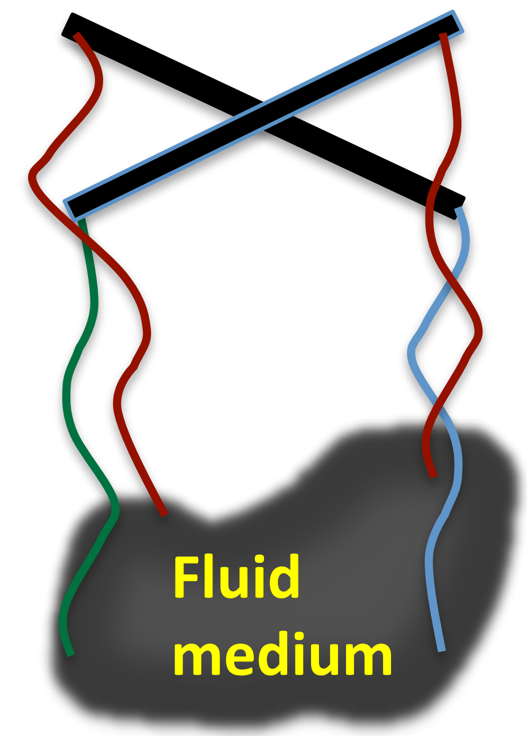
General approach:

1. Equations of motion (mass and momentum conservation)
2. Non-dimensionalisation
3. Identification of BASE-FLOW (laminar solution)
4. Decomposition of dependent variables into MEAN & FLUCTUATING quantities
5. Substitution into equations of motion
6. LINEARISATION (subtract base-flow equations; remove non-linear terms)
7. Reduce linearised equations to some compact form (often a single equation)
8. Express dependent variables in terms of NORMAL MODES
9. Introduction of normal modes into linearised equation:
   - PDE system becomes a single ODE, but with too many unknowns
10. Specify a value for one of the unknowns (wavenumber for instance), solve for others: generally an EIGENVALUE PROBLEM

Fluid motions constrained by:

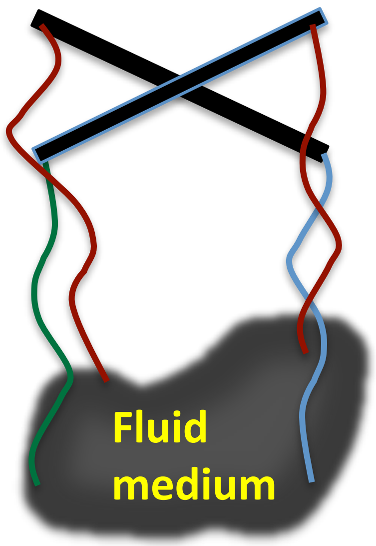
- Mass conservation
- Momentum conservation
- Boundary conditions

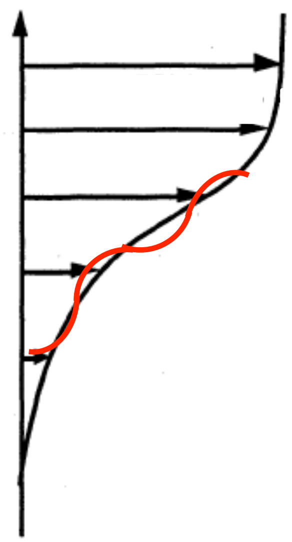




Linear dynamics in an incompressible, 2D fluid are governed by

$$
\begin{aligned}
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= 0 \\
\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y}\tilde{v} + \frac{\partial \tilde{p}}{\partial x} &= \mathrm{Re}^{-1}\nabla^2\tilde{u} \\
\frac{\partial \tilde{v}}{\partial t} + U\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} &= \mathrm{Re}^{-1}\nabla^2\tilde{v}
\end{aligned}
$$





From here we can derive the
Orr-Sommerfeld & Rayleigh equations

(1)
$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0$$

(2)
$$\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y}\tilde{v} + \frac{\partial \tilde{p}}{\partial x} = \mathrm{Re}^{-1}\nabla^2\tilde{u}$$

(3)
$$\frac{\partial \tilde{v}}{\partial t} + U\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} = \mathrm{Re}^{-1}\nabla^2\tilde{v}$$

Take divergence of momentum equations, $\frac{\partial}{\partial x}(2)+\frac{\partial}{\partial y}(3)$.

$$\frac{\partial}{\partial t}\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right) + U\frac{\partial}{\partial x}\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right) + 2\frac{\mathrm{d}U}{\mathrm{d}y}\frac{\partial \tilde{v}}{\partial x} + \nabla^2\tilde{p} = \mathrm{Re}^{-1}\nabla^2\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \tilde{u}}{\partial x}+\frac{\partial \tilde{v}}{\partial y}\right)+U\frac{\partial}{\partial x}\left(\frac{\partial \tilde{u}}{\partial x}+\frac{\partial \tilde{v}}{\partial y}\right)+2\frac{\mathrm{d}U}{\mathrm{d}y}\frac{\partial \tilde{v}}{\partial x}+\nabla^2\tilde{p}=\mathrm{Re}^{-1}\nabla^2\left(\frac{\partial \tilde{u}}{\partial x}+\frac{\partial \tilde{v}}{\partial y}\right)$$

Use continuity equation to remove all terms expressing divergence of the fluctuation field

$$\nabla^2\tilde{p}=-2\frac{\mathrm{d}U}{\mathrm{d}y}\frac{\partial \tilde{v}}{\partial x},$$

This is the Poisson equation for pressure. In a compressible system it would take the form of a wave equation.

Now take the Laplacian of the y-momentum equation

$$\nabla^2\left(\frac{\partial \tilde{v}}{\partial t}\right) + \nabla^2\left(U\frac{\partial \tilde{v}}{\partial x}\right) + \nabla^2\left(\frac{\partial \tilde{p}}{\partial y}\right) = \nabla^2(\mathrm{Re}^{-1}\nabla^2\tilde{v}).$$

Using the identity $\nabla^2(fg) = g\nabla^2 f + f\nabla^2 g + 2\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + 2\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}$

to expand the second term, and substituting from the Poisson equation, the equation simplifies to,

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right]\nabla^2\tilde{v} - \frac{\mathrm{d}^2U}{\mathrm{d}y^2}\frac{\partial \tilde{v}}{\partial x} = (\mathrm{Re}^{-1}\nabla^4\tilde{v})$$

1. Recap. of lecture 2

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right]\nabla^2\tilde{v} - \frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\frac{\partial \tilde{v}}{\partial x} = (\mathrm{Re}^{-1}\nabla^4\tilde{v})$$

As the x- and t- directions are homogeneous, normal modes can be used,

$$\tilde{v}(x,y,t) = \hat{v}(y)\mathrm{e}^{i\alpha(x-ct)}$$

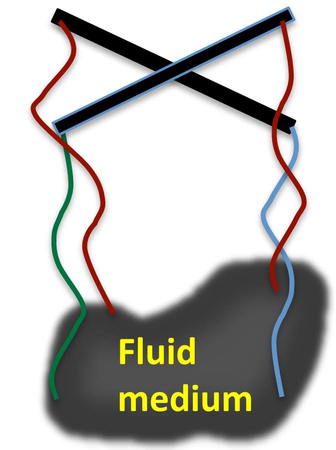
Fourier transforming the equation, with,

$$\frac{\partial \tilde{v}}{\partial x} \rightarrow i\alpha\hat{v}\mathrm{e}^{i\alpha(x-ct)}$$

$$\left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right] \rightarrow (-i\alpha c + Ui\alpha)$$

$$\nabla^2\tilde{v} \rightarrow \left[-\alpha^2 + \frac{\mathrm{d}^2}{\mathrm{d}y^2}\right]\hat{v}(y)\mathrm{e}^{i\alpha(x-ct)}$$

1. Recap. of lecture 2



gives the Orr-Sommerfeld equation,

$$(U-c)\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right]\hat{v}-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\hat{v}=\frac{1}{i\alpha\mathrm{Re}}\left[\frac{\mathrm{d}^4}{\mathrm{d}y^4}-2\alpha^2\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\alpha^4\right]\hat{v}$$

If we assume an inviscid fluid, this reduces to the Rayleigh equation,

$$(U-c)\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right]\hat{v}-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\hat{v}=0$$

The Squire transformation

3. The Squire transformation

Squire (1933) identified and exploited a similarity between the 2- and 3-D Orr-Sommerfeld equations,

Consider a 3-D disturbance, to a base flow, $U(y)$, with polar wavenumber,

$$\tilde{\alpha} = \sqrt{\alpha_{3D} + \beta_{3D}}$$

and which leads to an unstable solution of the 3-D Orr-Sommerfeld equation

$$(U - c)\left(\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2} - \tilde{\alpha}^2\mathbf{v}(y)\right) - \frac{\mathrm{d}^2U}{\mathrm{d}y^2}\mathbf{v}(y) = \frac{1}{i\alpha_{3D}\mathrm{Re}_{3D}}\left(\frac{\mathrm{d}^4\mathbf{v}(y)}{\mathrm{d}y^4} - 2\tilde{\alpha}^2\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2} + \tilde{\alpha}^4\mathbf{v}(y)\right)$$

$$(U-c)\left(\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2} - \tilde{\alpha}^2\mathbf{v}(y)\right) - \frac{\mathrm{d}^2U}{\mathrm{d}y^2}\mathbf{v}(y) = \frac{1}{i\alpha_{3D}\mathrm{Re}_{3D}}\left(\frac{\mathrm{d}^4\mathbf{v}(y)}{\mathrm{d}y^4} - 2\tilde{\alpha}^2\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2} + \tilde{\alpha}^4\mathbf{v}(y)\right)$$

Compare with 2-D Orr-Sommerfeld equation

$$(U-c)\left(\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2} - {\alpha_{2D}}^2\mathbf{v}(y)\right) - \frac{\mathrm{d}^2U}{\mathrm{d}y^2}\mathbf{v}(y) = \frac{1}{i\alpha_{2D}\mathrm{Re}_{2D}}\left(\frac{\mathrm{d}^4\mathbf{v}(y)}{\mathrm{d}y^4} - 2{\alpha_{2D}}^2\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2} + {\alpha_{2D}}^4\mathbf{v}(y)\right)$$

These equations have identical solutions if:

$$\alpha_{2D} = \tilde{\alpha} = \sqrt{\alpha_{3D} + \beta_{3D}}$$

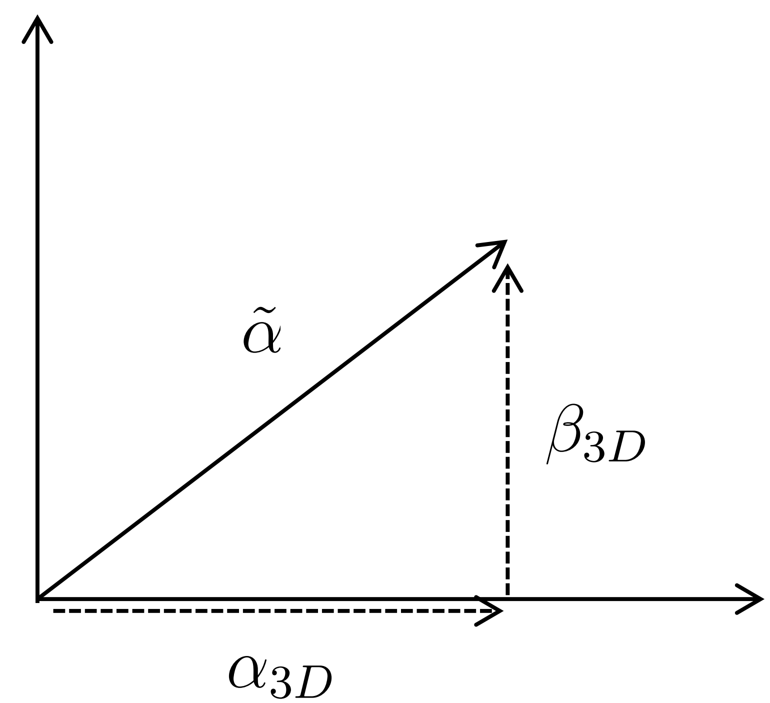
$$\alpha_{2D}\mathrm{Re}_{2D} = \alpha_{3D}\mathrm{Re}_{3D}$$

$$\mathrm{Re}_{2D} = \frac{\alpha_{3D}}{\alpha_{2D}}\mathrm{Re}_{3D} = \frac{\alpha_{3D}}{\tilde{\alpha}}\mathrm{Re}_{3D}$$

3. The Squire transformation

$$\alpha_{2D} = \tilde{\alpha} = \sqrt{\alpha_{3D} + \beta_{3D}}$$

$$\alpha_{2D}\mathrm{Re}_{2D} = \alpha_{3D}\mathrm{Re}_{3D}$$
$$\mathrm{Re}_{2D} = \frac{\alpha_{3D}}{\alpha_{2D}}\mathrm{Re}_{3D} = \frac{\alpha_{3D}}{\tilde{\alpha}}\mathrm{Re}_{3D}$$



For any unstable 3D disturbance

$$\tilde{\alpha} = \sqrt{\alpha_{3D} + \beta_{3D}}$$

There exists an unstable 2D disturbance

$\alpha_{2D} = \tilde{\alpha}$ **at** $\mathrm{Re}_{2D} = \frac{\alpha_{3D}}{\tilde{\alpha}}\mathrm{Re}_{3D}$

Squire's theorem: If an exact two-dimensional parallel flow admits an unstable 3-D disturbance for a certain value of the Reynolds number, it also admits an unstable 2-D disturbance at a lower Reynolds number

OR

Squire's theorem: To each unstable 3-D disturbance there corresponds a more unstable 2-D disturbance

OR

Squire's theorem: To obtain the minimum critical Reynolds number it is sufficient to consider only two-dimensional disturbances

2. Solving eigenvalue problems

Orr-Sommerfeld equation

$$(U-c)\Big(\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2}-\alpha^2\mathbf{v}(y)\Big)-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\mathbf{v}(y)=\frac{1}{i\alpha\mathrm{Re}}\Big(\frac{\mathrm{d}^4\mathbf{v}(y)}{\mathrm{d}y^4}-2\alpha^2\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2}+\alpha^4\mathbf{v}(y)\Big)$$

Fourth-order, 4 boundary conditions

Rayleigh equation

$$(U-c)\Big(\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2}-\alpha^2\mathbf{v}(y)\Big)-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\mathbf{v}(y)=0$$

Second-order, 2 boundary conditions

At rigid wall:

v=0 - Rayleigh & O-S
dv/dy=0 (since u=du/dx=0 & continuity) - only O-S

At infinity:

v & dv/dy both bounded (numerically: zero for large y)

Orr-Sommerfeld equation

$$(U-c)\left(\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2}-\alpha^2\mathbf{v}(y)\right)-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\mathbf{v}(y)=\frac{1}{i\alpha\mathrm{Re}}\left(\frac{\mathrm{d}^4\mathbf{v}(y)}{\mathrm{d}y^4}-2\alpha^2\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2}+\alpha^4\mathbf{v}(y)\right)$$

Rearrange as

$$\left[U\left(\frac{d^2}{dy^2}-\alpha^2\right)-\frac{d^2U}{dy^2}-\frac{1}{i\alpha\,\mathrm{Re}}\left(\frac{d^4}{dy^4}-2\alpha^2\frac{d^2}{dy^2}+\alpha^4\right)\right]\mathbf{v}=c\left(\frac{d^2}{dy^2}-\alpha^2\right)\mathbf{v}$$

$$\mathrm{L}\mathbf{v}=c\mathrm{F}\mathbf{v}$$

Generalised eigenvalue problem with linear operators, L and F, and eigenvalue, c.

We must discretise the operators as matrices and then use Matlab routine, eig(L,F)

Consider a simpler problem first:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}\right]\mathbf{v} = \lambda(-1)\mathbf{v}$$

Appears in linear acoustics, vibration of membranes, heat transfer,…

With B.C. v=0 for y=1 & y=-1 equation has analytical solution

$$\lambda_n = n^2\frac{\pi^2}{4}, n = 1,2,3,\ldots$$

$$v_n = \cos\left(\frac{n\pi}{2}y\right) \quad \text{for odd } n$$

$$v_n = \sin\left(\frac{n\pi}{2}y\right) \quad \text{for even } n$$

Consider a simpler problem first:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}\right]\mathbf{v} = \lambda(-1)\mathbf{v}$$

We first need to construct a differentiation matrix

Could use:

- Finite differences (e.g. EVPExample_FiniteDifferences.m)
- Spectral methods (e.g. EVPExample_Chebyshev.m)

2. Solving eigenvalue problems

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}\right]\mathbf{v} = \lambda(-1)\mathbf{v}$$

Finite differences

v is sampled at *N* grid points, with spacing, Δy

For interior points:

$$\frac{\mathrm{d}^2\mathbf{v}}{\mathrm{d}y^2}(y_i) \approx \frac{\mathbf{v}(y_{i+1}) - 2\mathbf{v}(y_i) + \mathbf{v}(y_{i-1})}{\Delta y^2} + O(\Delta y^2)$$

For boundary points:

$$\frac{\mathrm{d}^2\mathbf{v}}{\mathrm{d}y^2}(y_i) \approx \frac{\mathbf{v}(y_{i+2}) - 2\mathbf{v}(y_{i+1}) + \mathbf{v}(y_i)}{\Delta y^2} + O(\Delta y)$$

$$D = \frac{1}{\Delta y^2}\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

Derivative is estimated from values of function in the neighbourhood of *y_i*

Differentiation matrix is sparse

Freedom to choose the grid

Polynomial interpolants: appropriate for bounded and/or non-periodic domains, as opposed to trigonometric interpolants, suitable for periodic domains.

Procedure:

- Use an Nth-order polynomial to represent the discrete data,
- Derivative of discrete data expressed, at each grid point, in terms of (analytical) derivatives of the interpolant,
- This provides a set of polynomial coefficients for each grid point
- The derivatives at each grid point is a function of all other grid points,
- Differentiation can be expressed in matrix form; the matrix is full on account of previous point

Polynomial interpolant

$$p_N(x) = \sum_{i=0}^{N} v(x_i) C_i(x)$$

Lagrange interpolation formula

Cardinal functions, $C_i(x)$, are polynomials of degree N that satisfy the condition,

$$C_i(x_j) = \delta_{ij}$$

and are defined by,

$$C_i(x) = \prod_{j=0, j\neq i}^{N} \frac{x - x_j}{x_i - x_j}$$

Pseudospectral method: Chebyshev polynomials

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad n \geq 1$$

→ **Orthogonal basis in [-1,1]**

Cardinal functions

$$\phi_j(x) = \frac{(-1)^j}{c_j} \frac{1-x^2}{(N-1)^2} \frac{T'_{N-1}(x)}{x - x_j}$$

Expansion

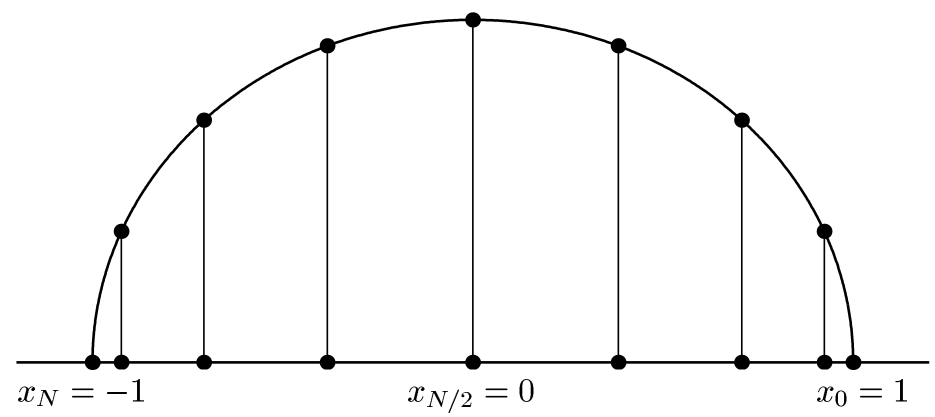
$$v(x) \approx p_{N-1}(x) = \sum_{j=1}^{N} \phi_j(x) v_j$$

Derivative

$$\frac{\mathrm{d}}{\mathrm{d}x} v(x) \approx \frac{\mathrm{d}}{\mathrm{d}x} p_{N-1}(x) = \sum_{j=1}^{N} \frac{\mathrm{d}}{\mathrm{d}x} \phi_j(x) v_j$$

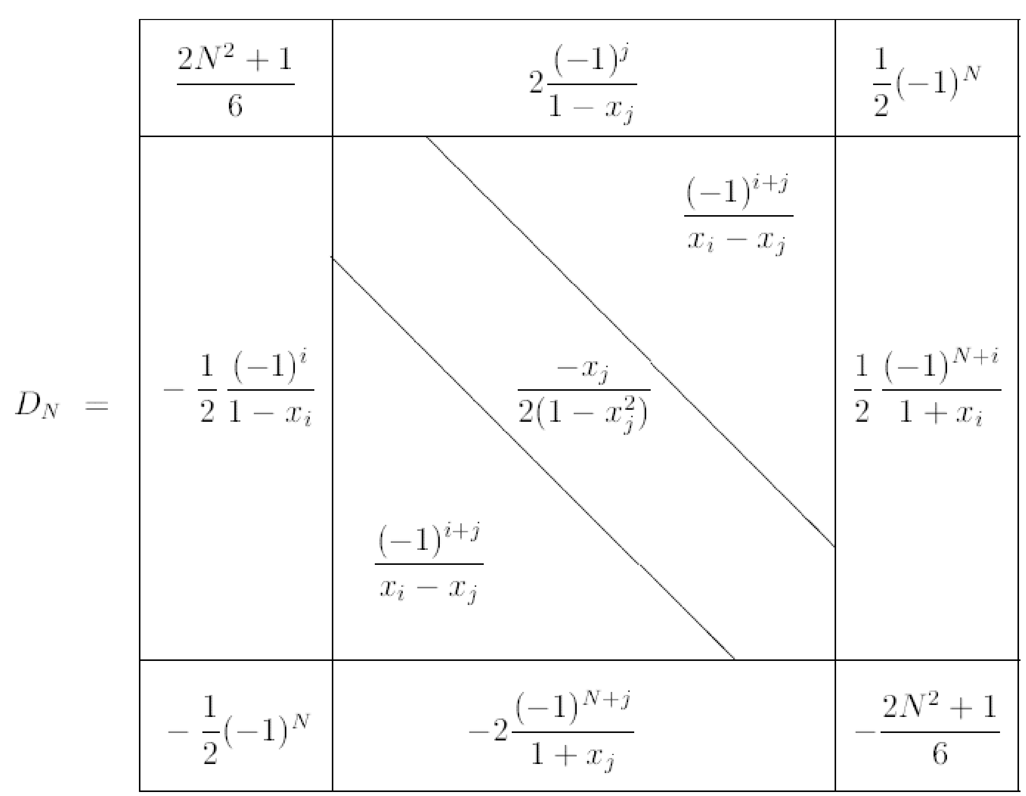
Pseudo-spectral method: Chebyshev polynomials

Chebyshev grid



Potential theory
cf. Trefethen 2001

Differentiation grid

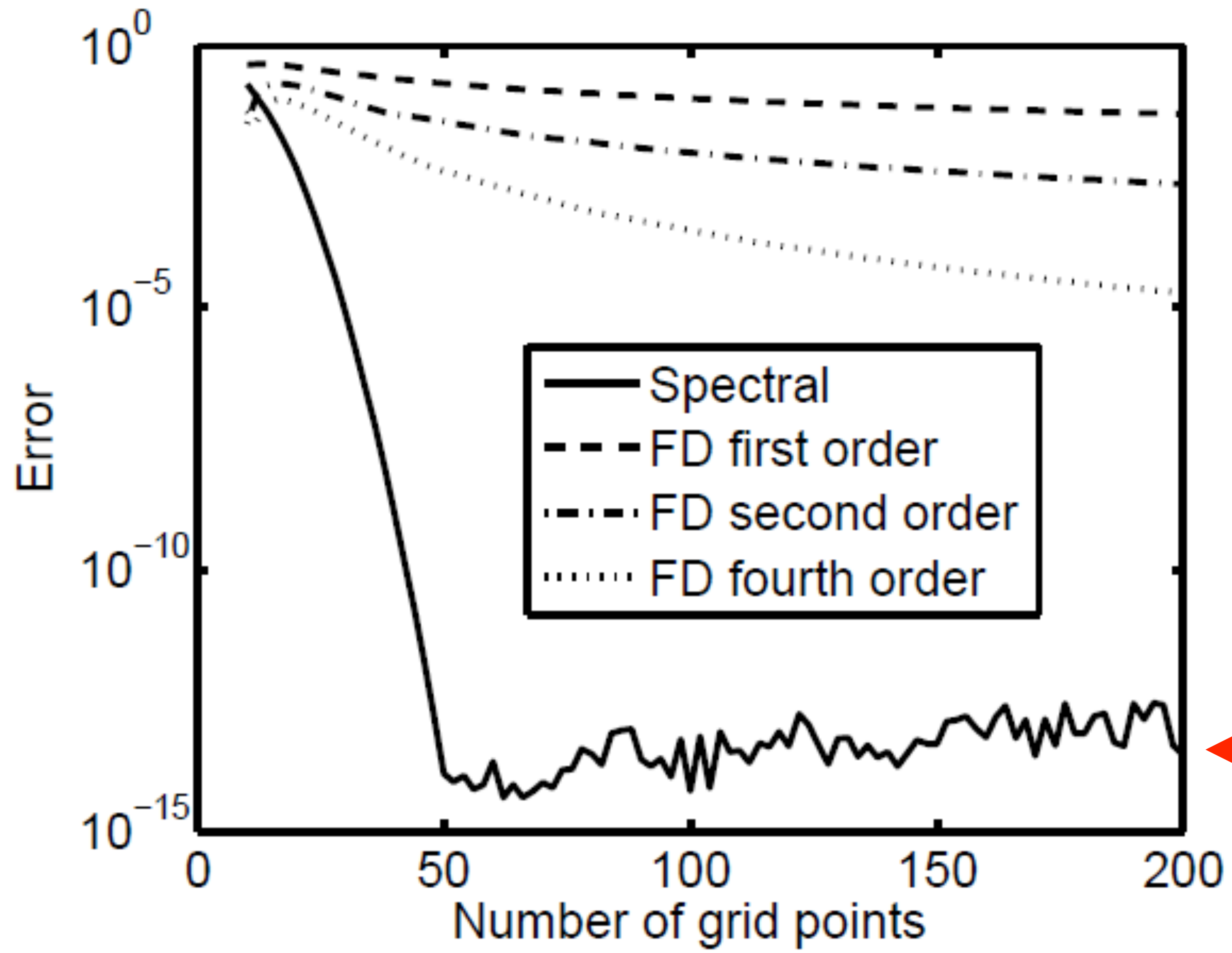


Differentiation matrix is full

Grid must comprise Chebyshev points

Finite-difference versus pseudo-spectral method

Error in calculation of e^{x^2}



Finite difference:
- **algebraic convergence**

Pseudo spectral:
- **exponential convergence**

Machine precision

See chebyshev_introduction.m

Example problem

- **Solve:**

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}\right]\mathbf{v} = \lambda(-1)\mathbf{v}$$

Compare with analytical solutions:

$$\lambda_n = n^2\frac{\pi^2}{4}, n = 1,2,3,\ldots$$

$$v_n = \cos\left(\frac{n\pi}{2}y\right) \quad \text{for odd } n$$

$$v_n = \sin\left(\frac{n\pi}{2}y\right) \quad \text{for even } n$$

Solution recipe

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}\right]\mathbf{v} = \lambda(-1)\mathbf{v} \longrightarrow \mathrm{L}\mathbf{v} = c\mathrm{F}\mathbf{v}$$

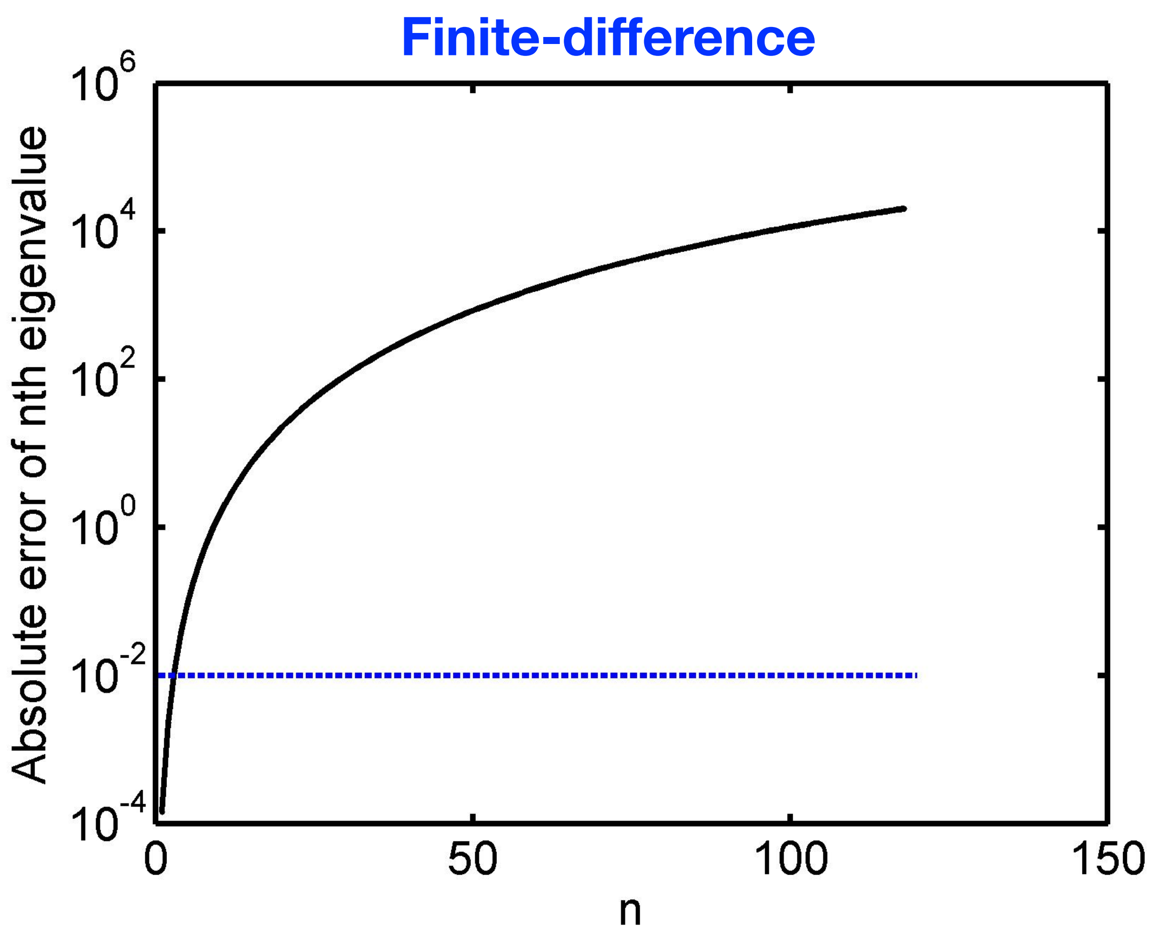
1. Define Chebyshev order and construct grid: [D,y]=cheb(N)

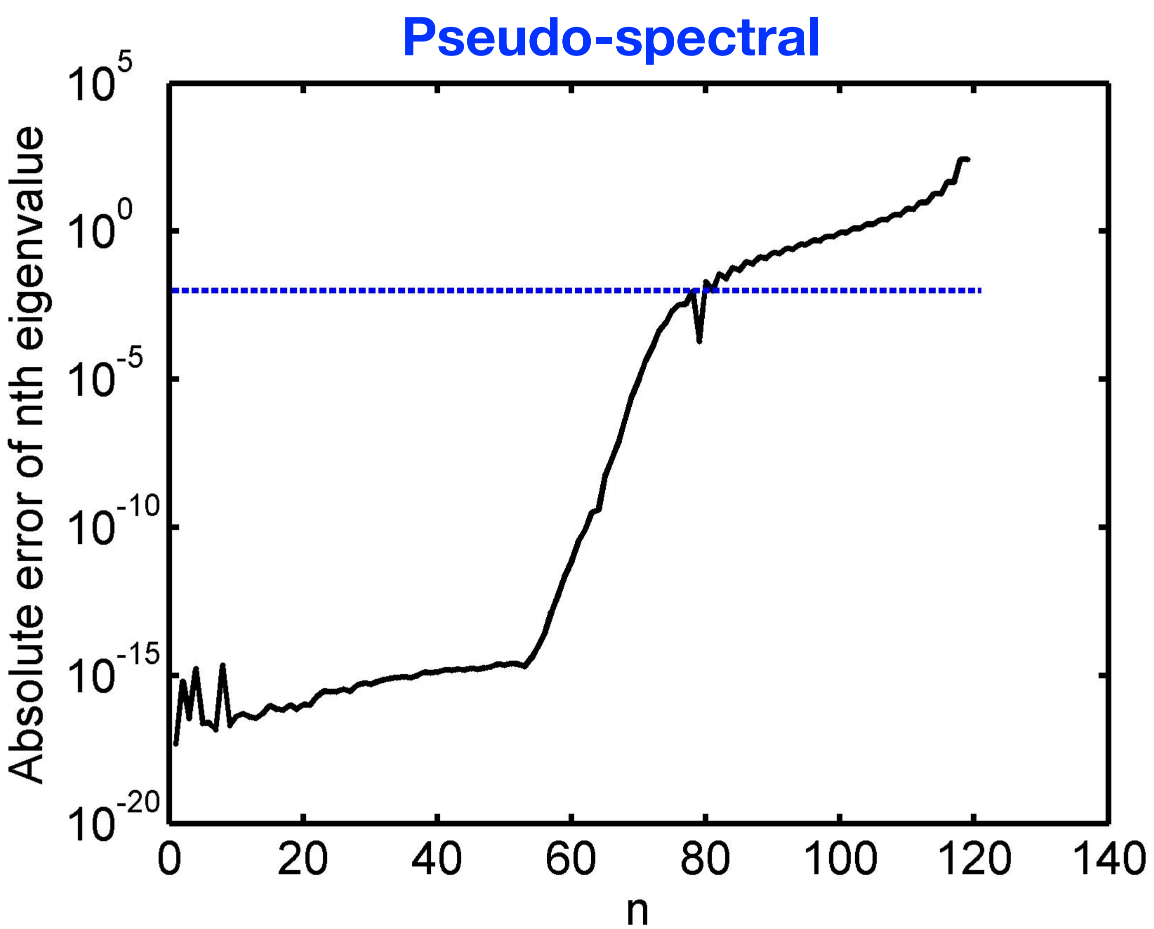
2. Set up eigenvalue problem: construct operators L **and** F **and impose boundary conditions** :

```
L(1,:)=0 ; L(1,1)=1 ; F(1,:)=0;
L(N+1,:)=0 ; L(N+1,N+1)=1 ; F(N+1,:)=0;
```

Solve eigenvalue problem: [V,lambda]=eig(L,F);

Finite-difference versus pseudo-spectral method





- Some eigenvalues (and eigenfunctions) converged, but some (at least half) should not be trusted
- Pseudo-spectral method better suited for accurate predictions
- But, revival of finite-difference schemes due to sparse-matrix algorithms: can compensate poor accuracy with larger N (cf. Gennaro et al. 2013)

3. Inviscid temporal instability of the mixing layer

Rayleigh equation

$$\left[U\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right)-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\right]\mathbf{v} = c\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right]\mathbf{v}$$

$$\mathbf{Lv} = c\mathbf{Fv}$$

Exercise: adapt the code used to solve

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}\right]\mathbf{v} = \lambda(-1)\mathbf{v}$$

to solve the temporal stability problem for the Rayleigh equation for a mixing layer with base-flow, $U = 0.5(1+\tanh(y))$, and BCs, $v \to 0$ for $y \to \pm\infty$: numerically, $v = 0$ for $y = \pm H$ with $H \gg 1$. Start with $\alpha = 0.1$. Compare with Michalke (1964)

Numerical tips

diag(U) : diagonal matrix, main diagonal = vector U

II=eye(size(D)) : identity matrix, same size as matrix D

ZZ=zeros(size(D)) : matrix or zeros, same size as matrix D

H=20; y=y*H; D=D/H; D2=D2/(H^2) : stretches domain from [-1,1] to [-H,H]

A*B : standard matrix multiplication (N_rows A = N_columns B)

A.*B : element-to-element multiplcation (dimensions of A and B must be equal)

A^2 is A*A (A should be a square matrix)

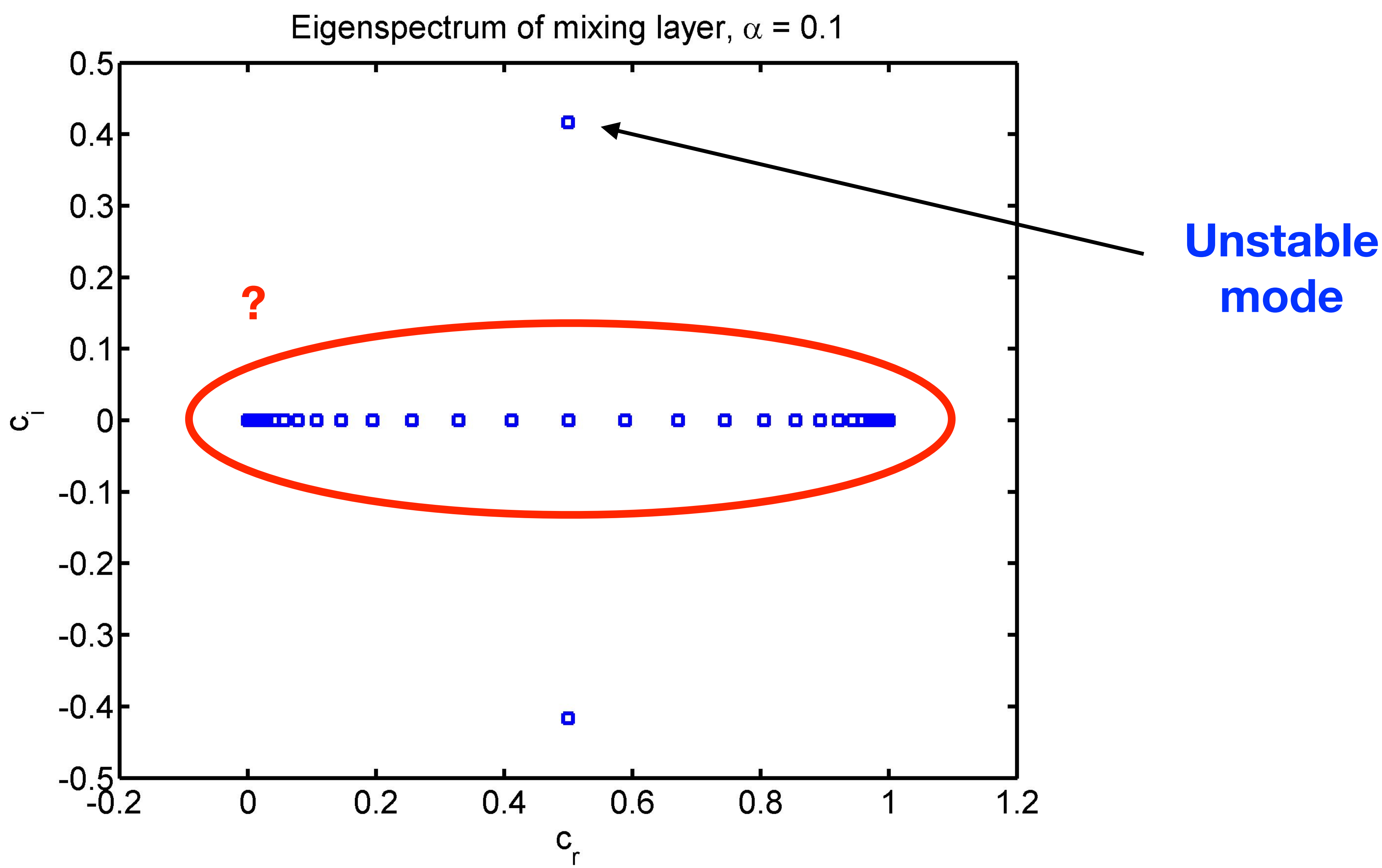
A.^2 is A.*A (A can have any size)

A(1,:) means all elements in row 1 of matrix A

A(:,5) means all elements in column 5 of matrix A

Expected results

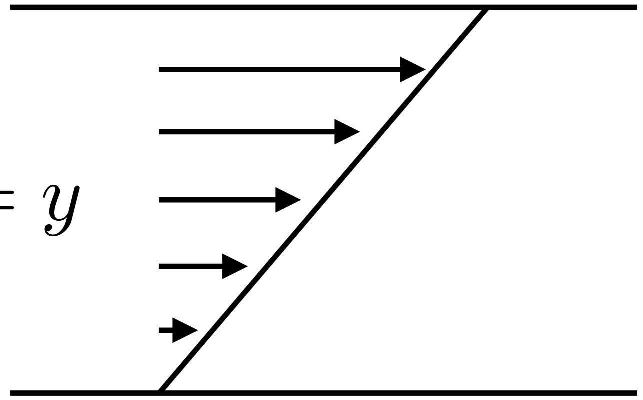
$$\left[U\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right)-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\right]\mathbf{v}=c\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right]\mathbf{v}$$



Critical layer and continuous spectrum

Consider the simplest shear flow: Couette flow

$$U(y) = y$$



Rayleigh's equation reduces to

$$(U - c)\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2\right)\mathbf{v} = 0$$

Critical layer and continuous spectrum

$$(U-c)\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right)\mathbf{v}=0$$

Two solution possibilities:

#1. $U \neq c \;\text{ for all }\; y \;\rightarrow \left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right)\mathbf{v}=0$

$$\mathbf{v}(y)=A\mathrm{e}^{\alpha y}+B\mathrm{e}^{-\alpha y}$$

But boundary conditions, $\mathbf{v}(y=0,1)=0$ **, imply that** $A=B=0$

$$\rightarrow \mathbf{v}(y)=0 \quad \textbf{!!}$$

Implying that fluctuations cannot exist in Couette flow: clearly not physical.

Critical layer and continuous spectrum

$$(U-c)\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right)\mathbf{v}=0$$

This implies solution possibility #2.

$$c=U(y_c)\quad \text{for}\quad 0\leq y_c\leq 1$$

which in turn implies that the coefficient of the highest derivative goes to zero: singularity.

Resolution of this problem if, when $U\neq c$ **,** $\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right)\mathbf{v}=0$

and, when $U=c$ **,** $\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right)\mathbf{v}\neq 0$

What function has this property?

Critical layer and continuous spectrum

The Dirac delta function, $\delta(y - y_c)$

So,

$$(U - c)\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2\right)\mathbf{v} = 0$$

can be solved only if,

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y} - \alpha^2\right)\mathbf{v}(y) = \delta(y - y_c)$$

whose solution is the impulse response (Green's function) of the problem, often written,

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y} - \alpha^2\right)G(y, y_c) = \delta(y - y_c)$$

Critical layer and continuous spectrum

The solution strategy: obtain and match solutions on either side of $y = y_c$

$$\mathbf{v}(y) = A\mathrm{e}^{\alpha y} + B\mathrm{e}^{-\alpha y}, \quad \text{for} \quad y < y_c$$
$$\mathbf{v}(y) = C\mathrm{e}^{\alpha y} + D\mathrm{e}^{-\alpha y}, \quad \text{for} \quad y > y_c$$

Boundary, matching and jump conditions

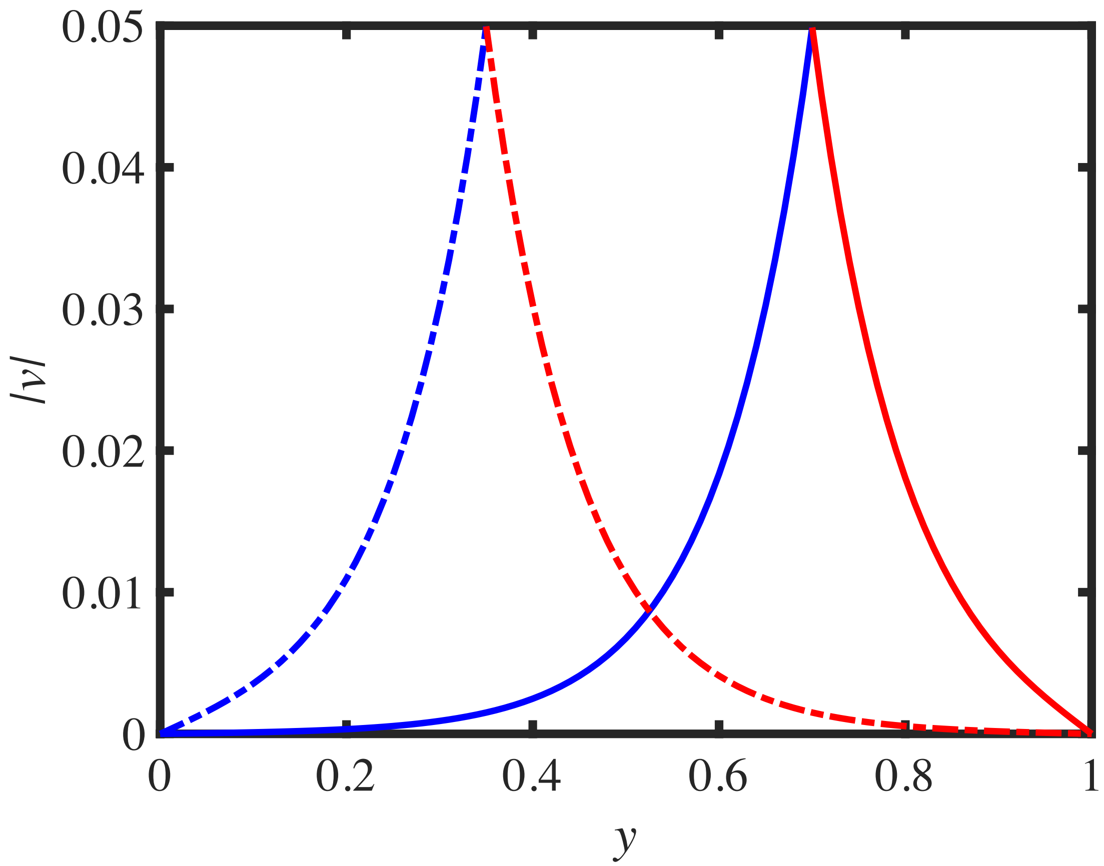
$$\mathbf{v}(0) = \mathbf{v}(1) = 0,$$
$$\mathbf{v}(y_c^+) = \mathbf{v}(y_c^-),$$
$$\frac{\partial \mathbf{v}}{\partial y}(y_c^+) - \frac{\partial \mathbf{v}}{\partial y}(y_c^-) = 1$$

Critical layer and continuous spectrum

Which has solution

$$\mathbf{v}(y) = -\frac{1}{\alpha \sinh(\alpha)} \left[\sinh(\alpha y) \sinh(\alpha(1-y_c)) \right] \quad \text{for} \quad y < y_c,$$

$$\mathbf{v}(y) = -\frac{1}{\alpha \sinh(\alpha)} \left[\sinh(\alpha y_c) \sinh(\alpha(1-y)) \right] \quad \text{for} \quad y > y_c,$$



Critical layer and continuous spectrum

In summary

In plane Couette flow, where linear dynamics are governed by,

$$(U-c)\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right)\mathbf{v}=0$$

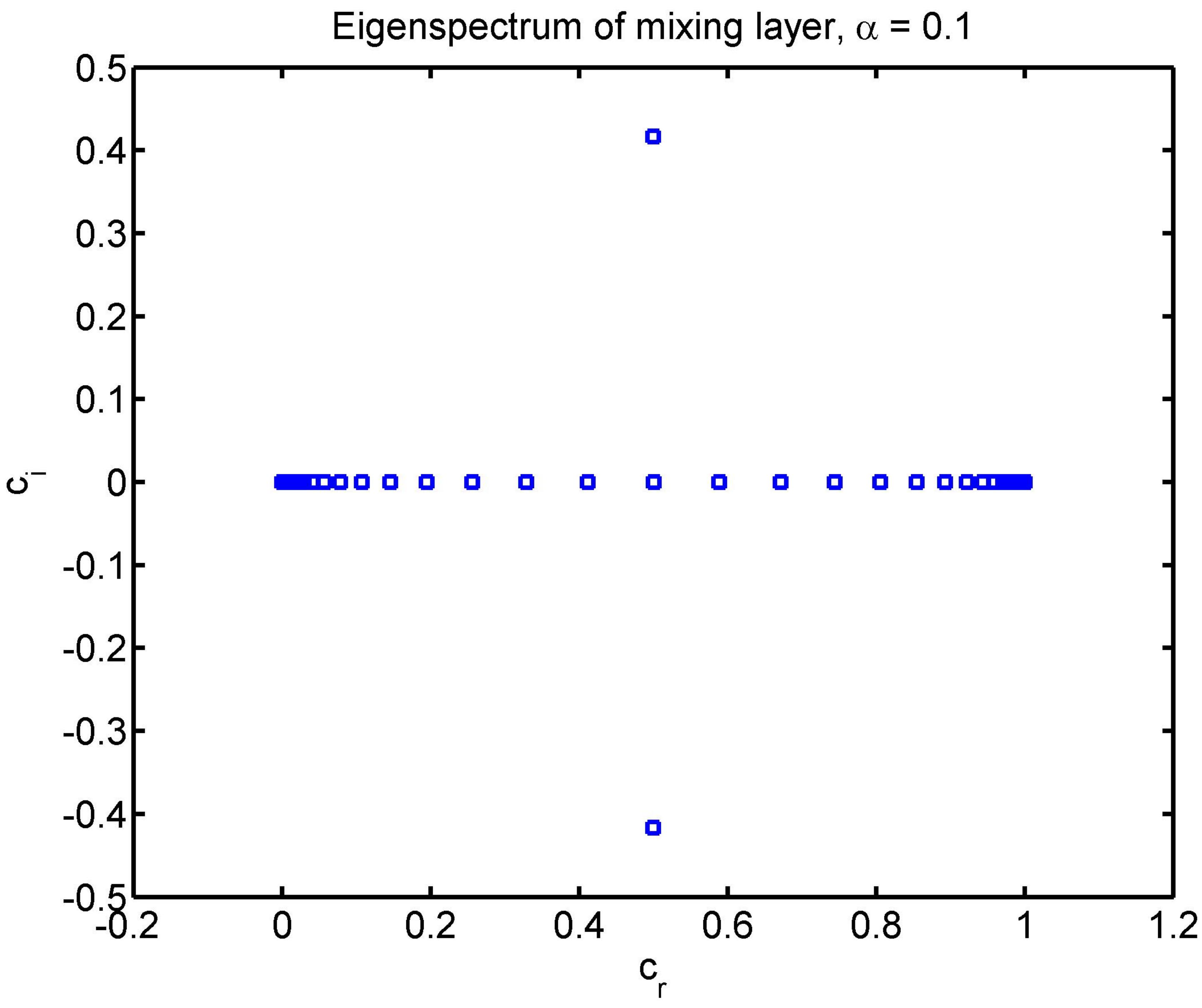
solutions can only be found by considering perturbations with phase velocity equal to the local base-flow velocity at the critical-layer position, $y=y_c$

The singularity that this entails leads to a continuum of impulse-response solutions at critical-point positions $0\leq y_c\leq 1$

These solutions form a continuous spectrum of critical-layer eigenvalues.

Critical layer an continuous spectrum

$$\left[U\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right)-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\right]\mathbf{v}=c\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right]\mathbf{v}$$



Expected results

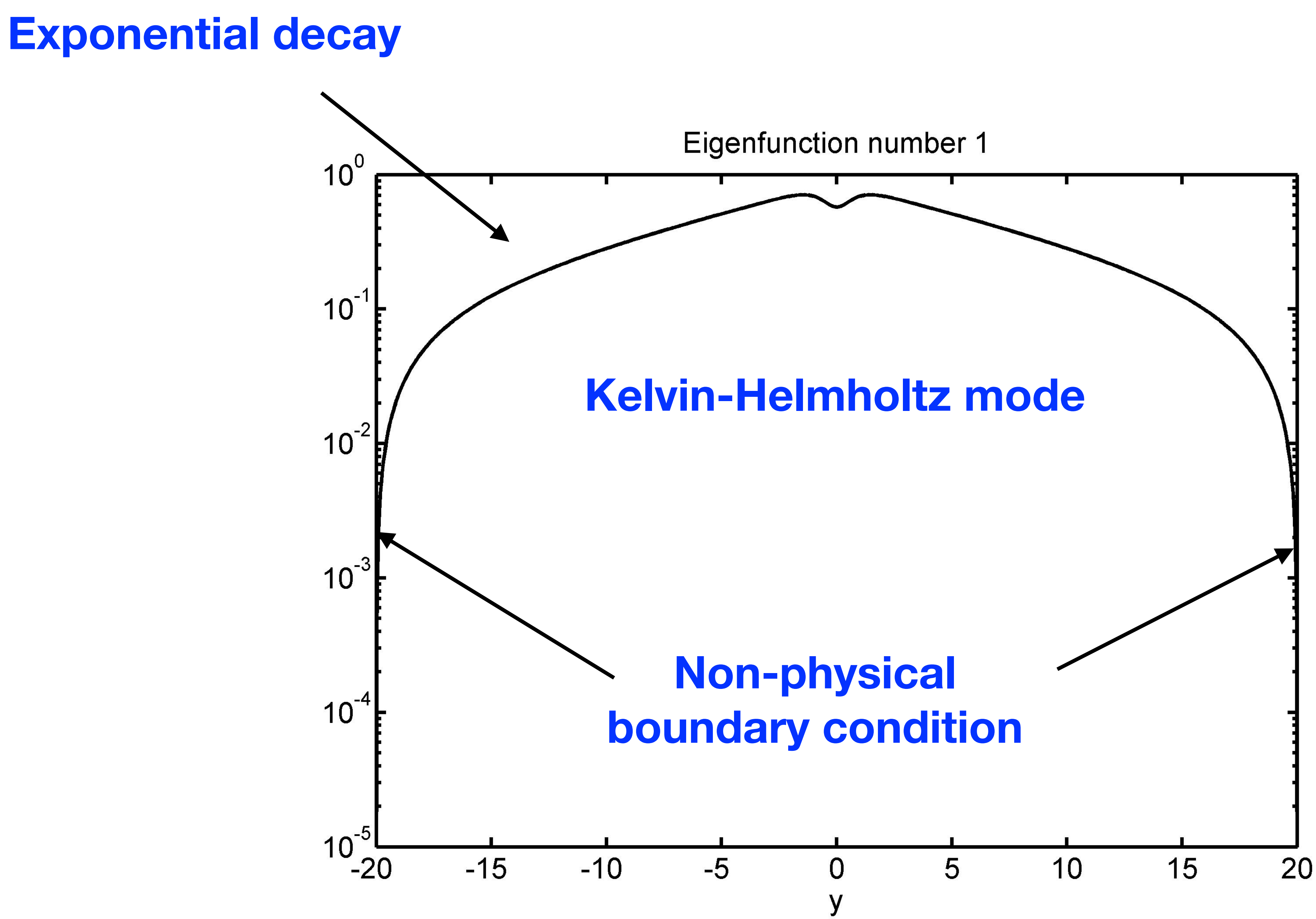
$$\left[U\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right)-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\right]\mathbf{v}=c\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right]\mathbf{v}$$

The eigenvalue-eigenfunction pair (c,$\mathbf{v}$) is a solution of the Rayleigh equation. The complex-conjugate pair, (c^*,$\mathbf{v}^*$), are also solutions,

$$\left[U\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right)-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\right]\mathbf{v}^*=c^*\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right]\mathbf{v}^*$$

The same is not true of the Orr-Sommerfeld equation, which is not self-adjoint on account of the viscous term.

Expected results (absolute value of eigenfunction associated with unstable eigenvalue)



Expected results

In region where *U* is constant, the Rayleigh equation,

$$\left[U\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right)-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\right]\mathbf{v}=c\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right]\mathbf{v}$$

reduces to,

$$(U-c)\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right)\mathbf{v}=0$$

if $c \neq U$

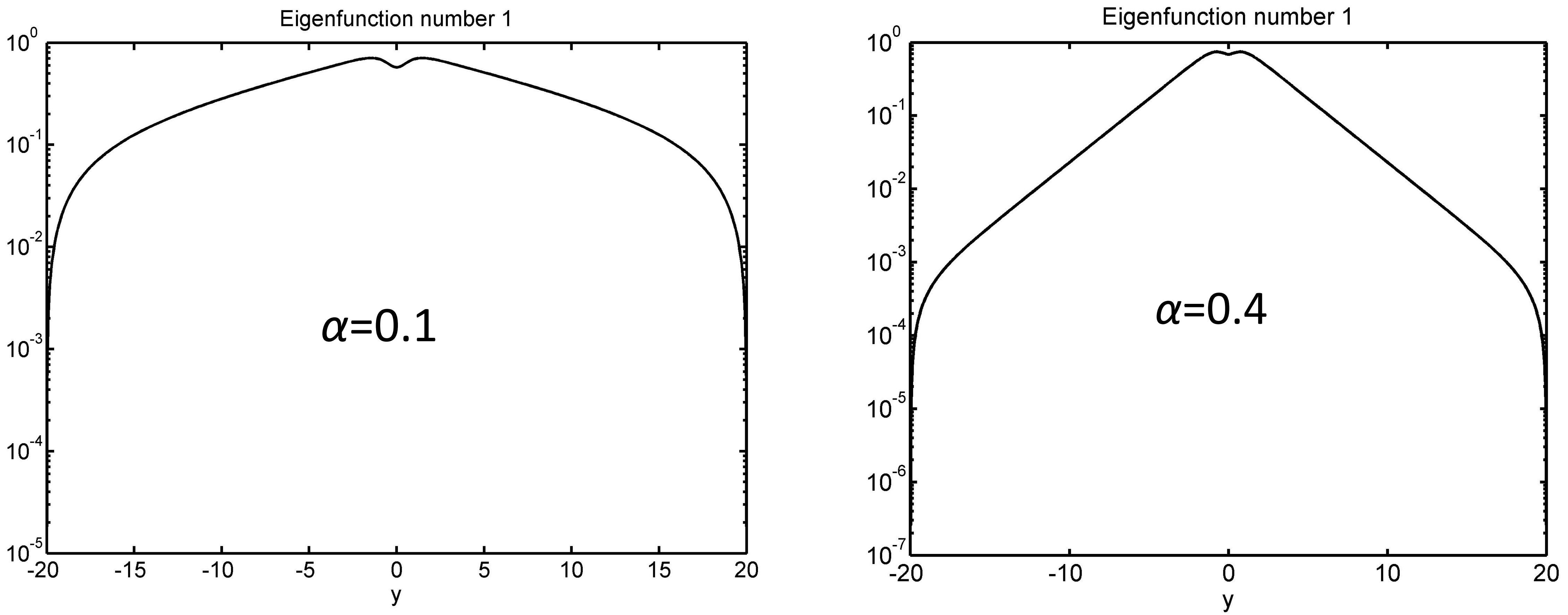
$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\alpha^2\right)\mathbf{v}=0$$

$$\longrightarrow \quad \mathbf{v}\propto \mathrm{e}^{\pm\alpha y}$$

plus and minus signs chosen such that solution be bounded at infinity

3. Inviscid temporal stability of the mixing layer

Expected results

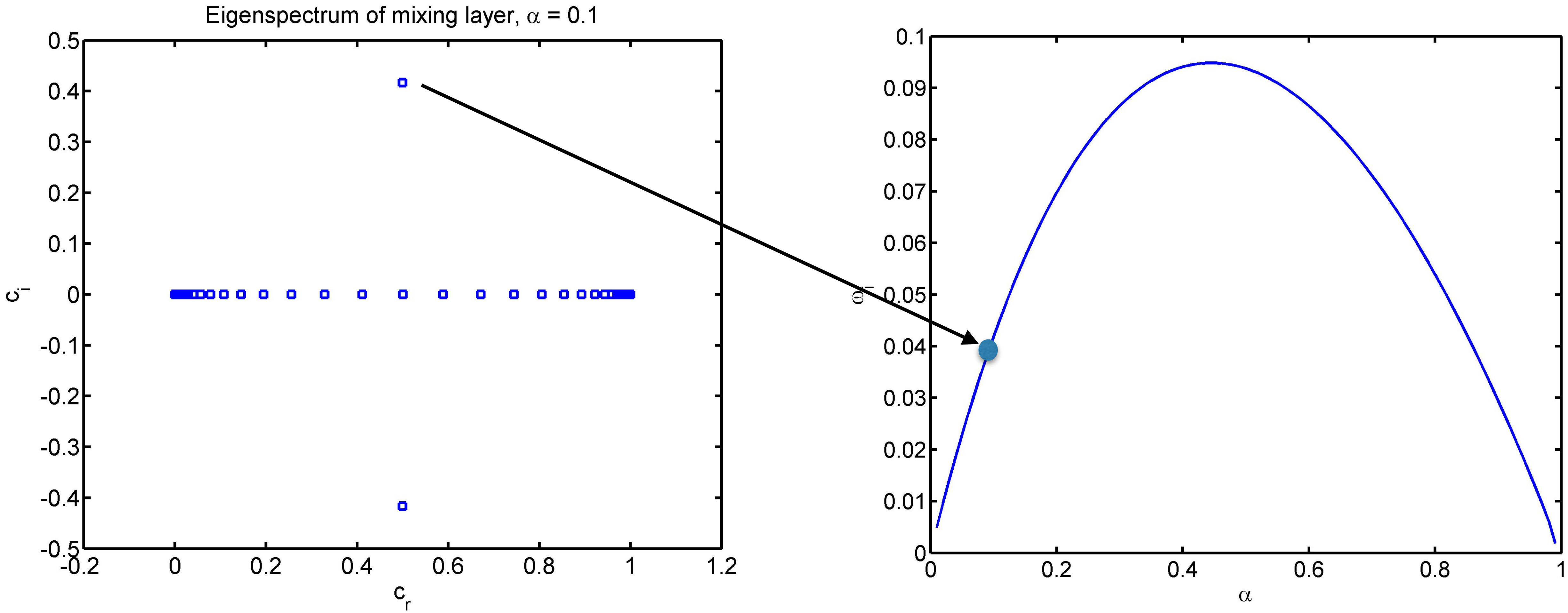


$\mathbf{v} \propto \mathrm{e}^{\pm \alpha y}$ → **Non-physical boundary condition more problematic at low wavenumber**

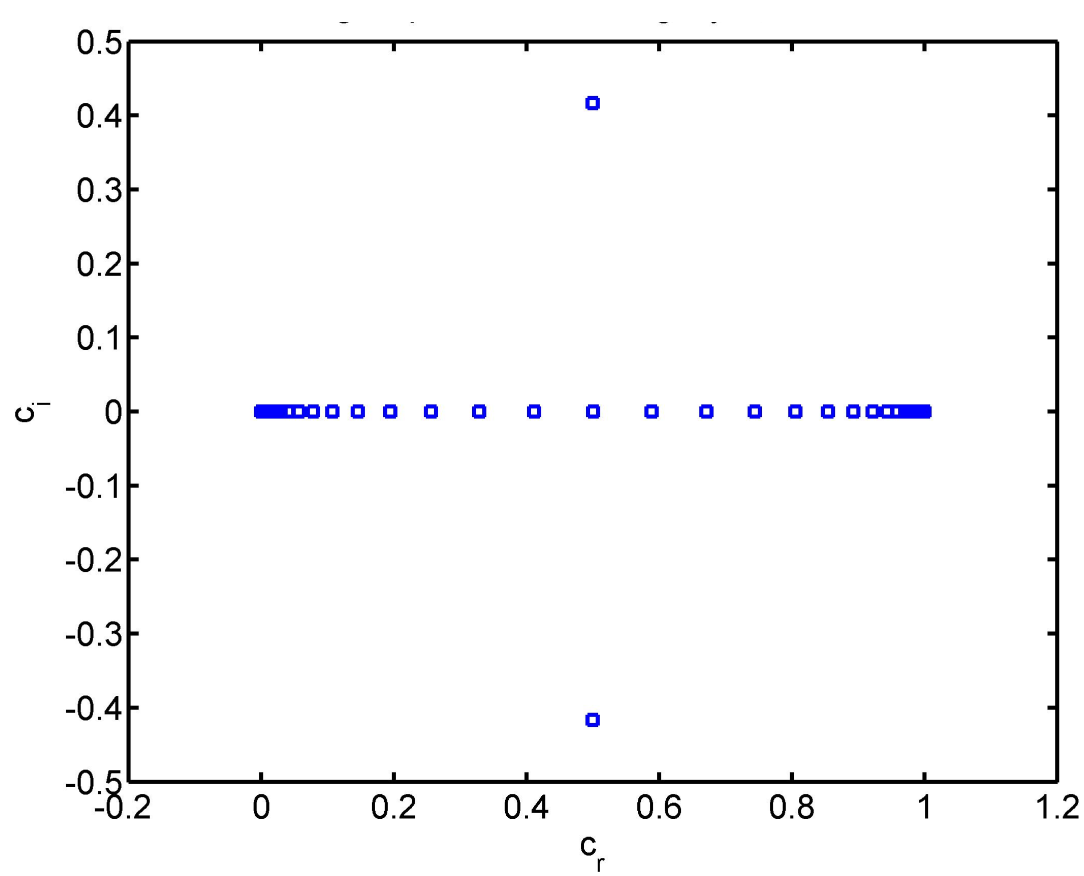
Domain size should be large compared to wavelength

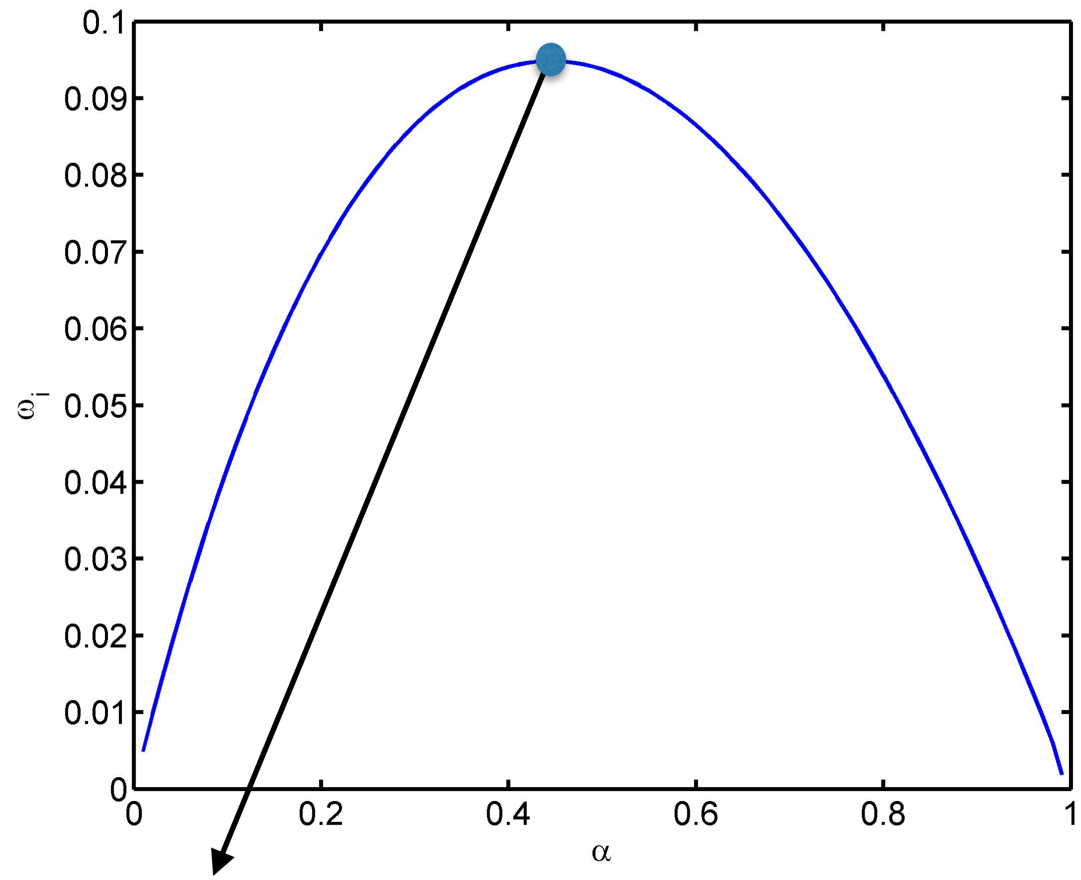
Expected results - most amplified wavenumber

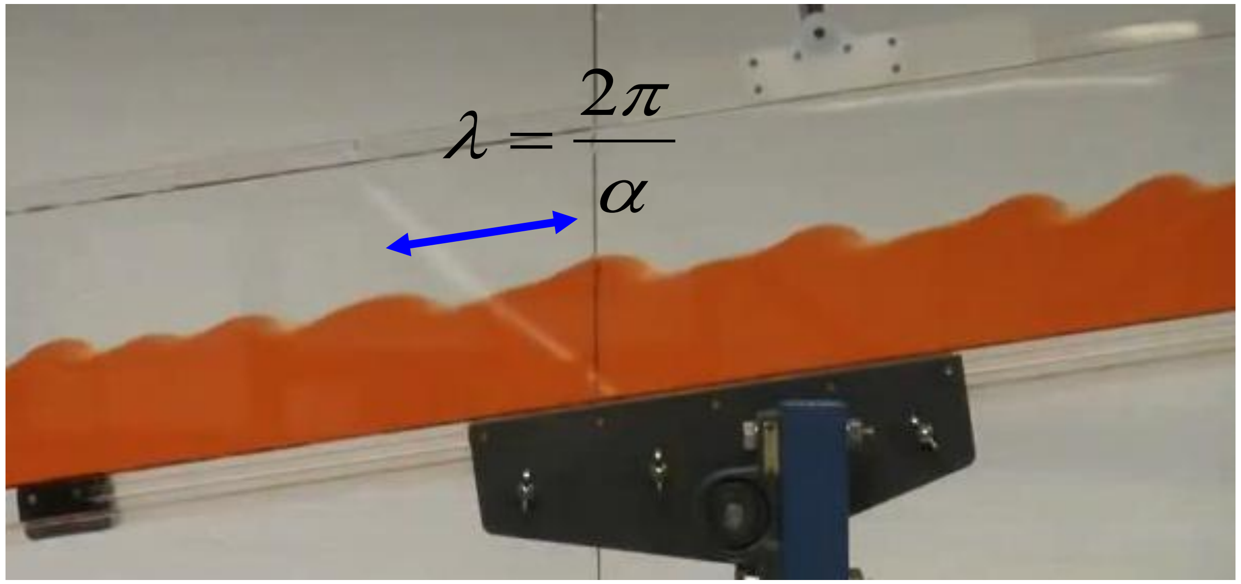
$$\tilde{v}(x,y,t) = \hat{v}\mathrm{e}^{i\alpha(x-ct)}$$
$$\omega_i = \alpha c_i$$



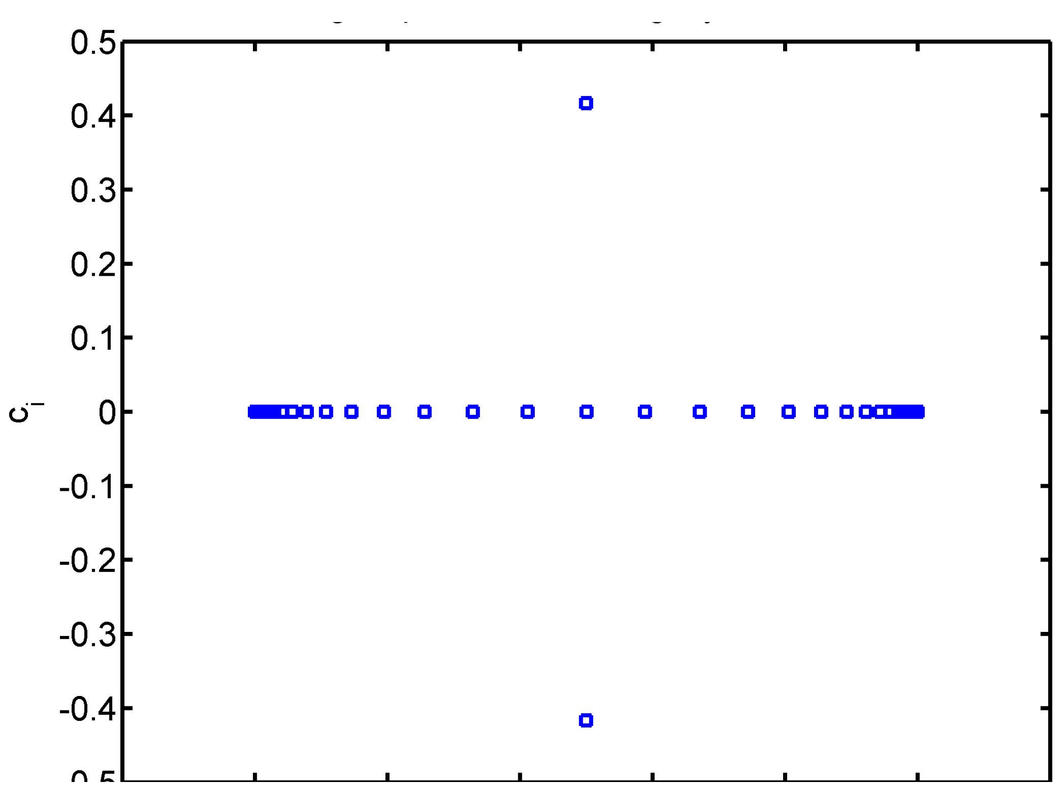
3. Inviscid temporal stability of the mixing layer

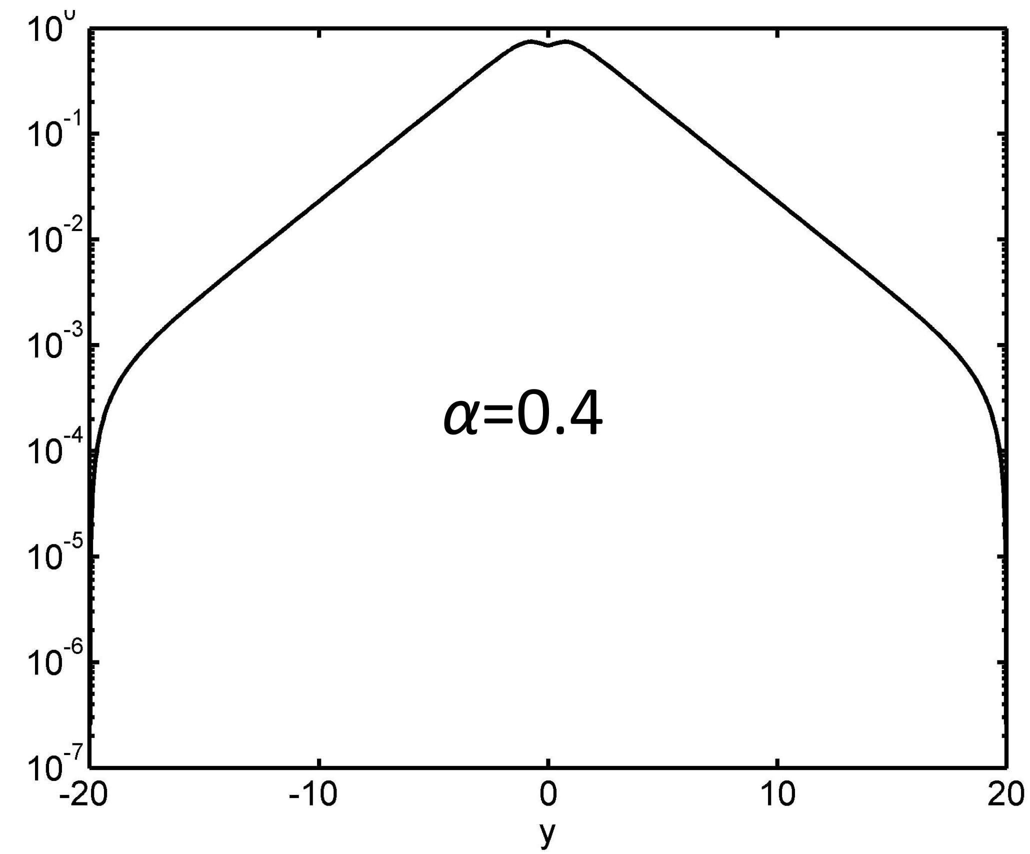


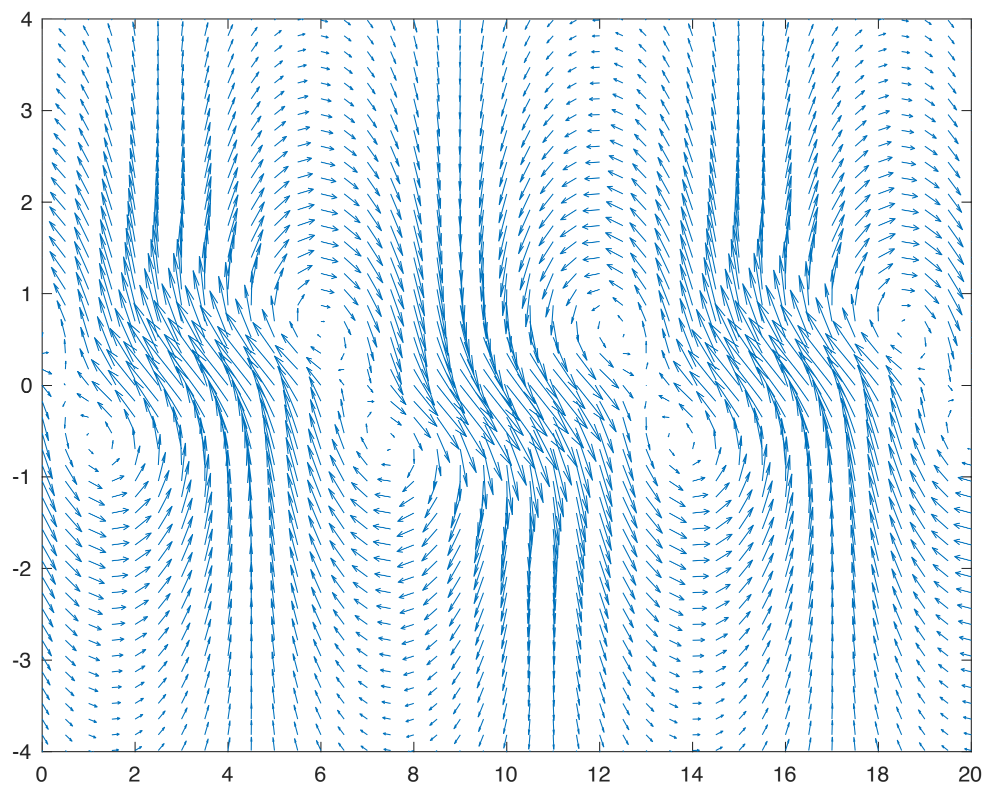


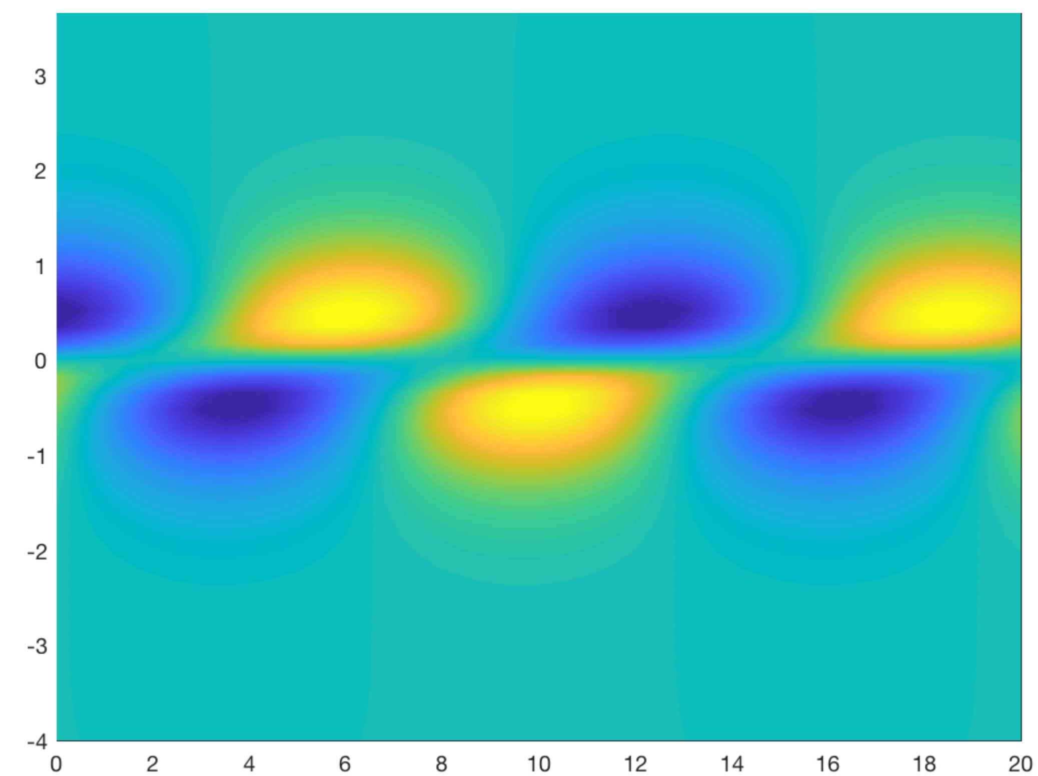


3. Inviscid temporal stability of the mixing layer



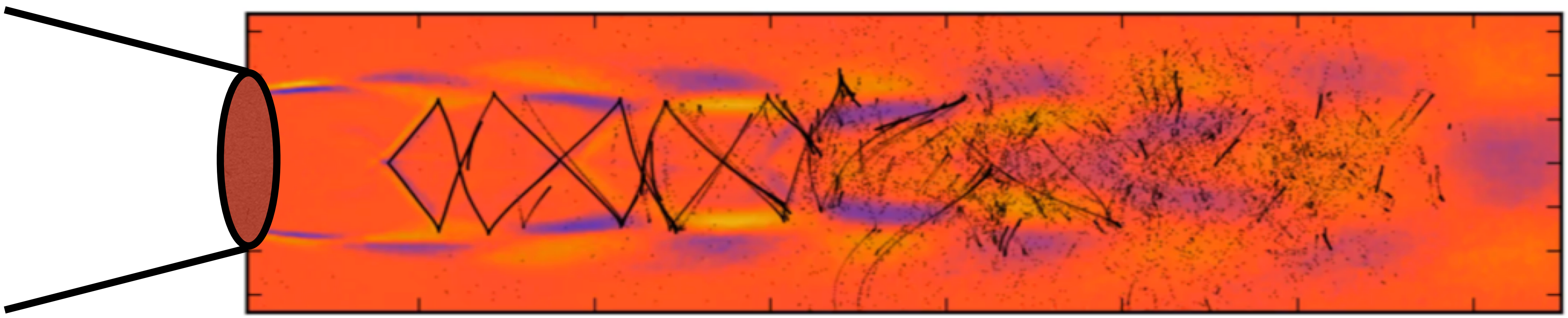


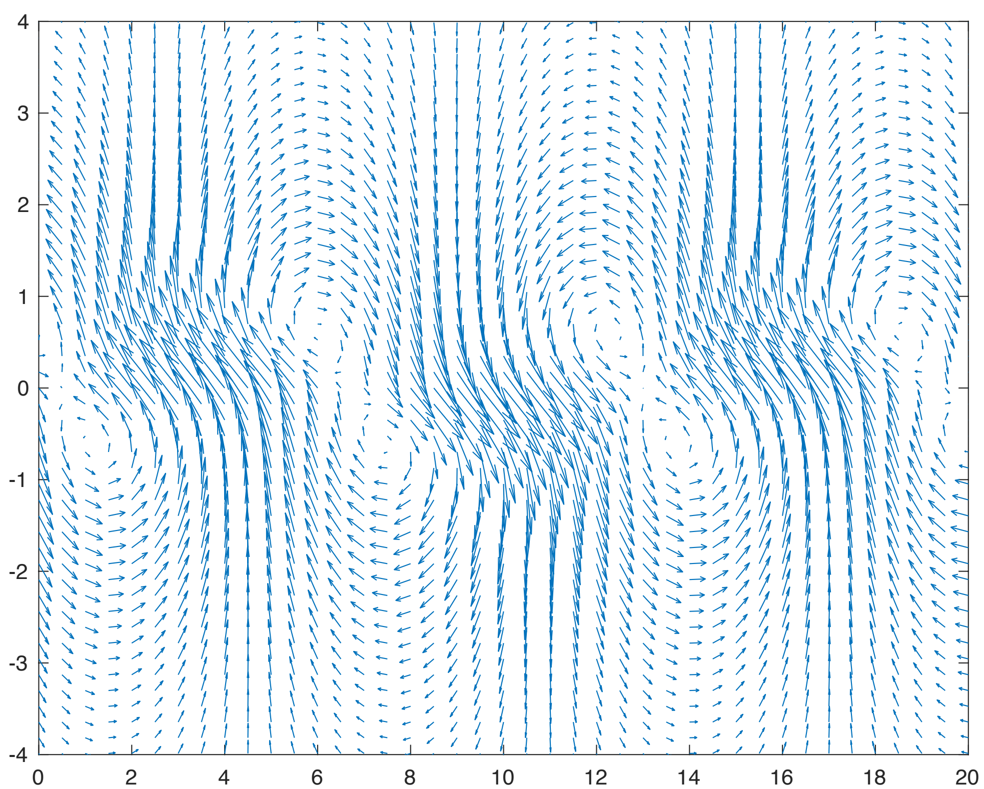


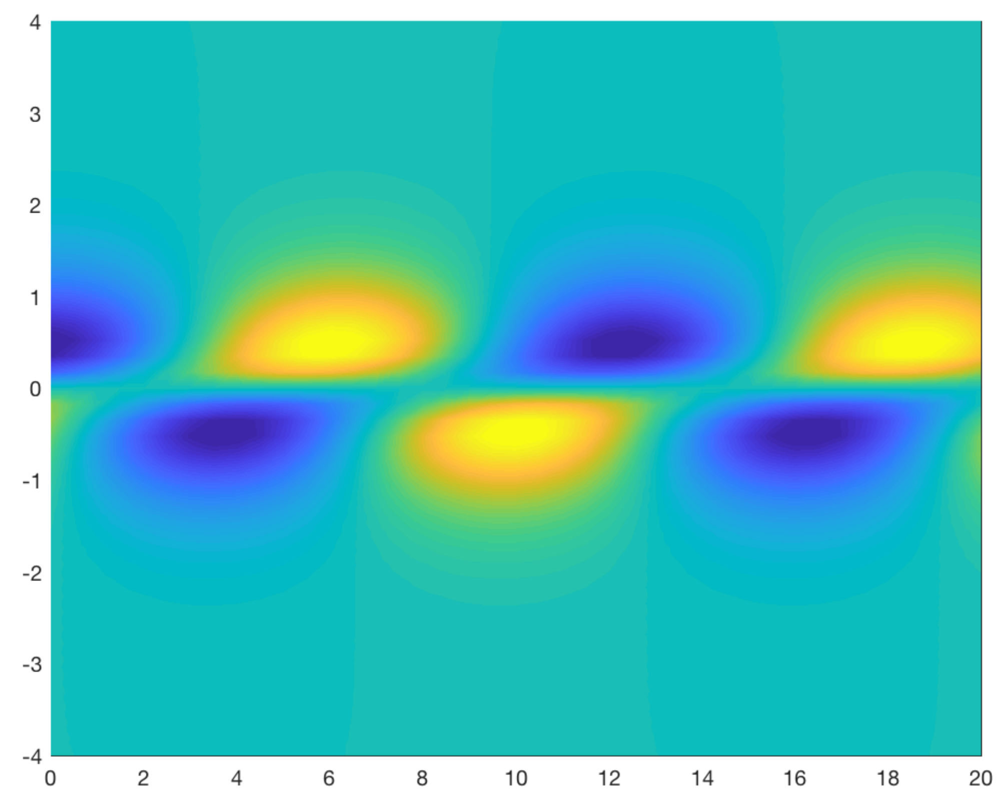


3. Inviscid temporal stability of the mixing layer

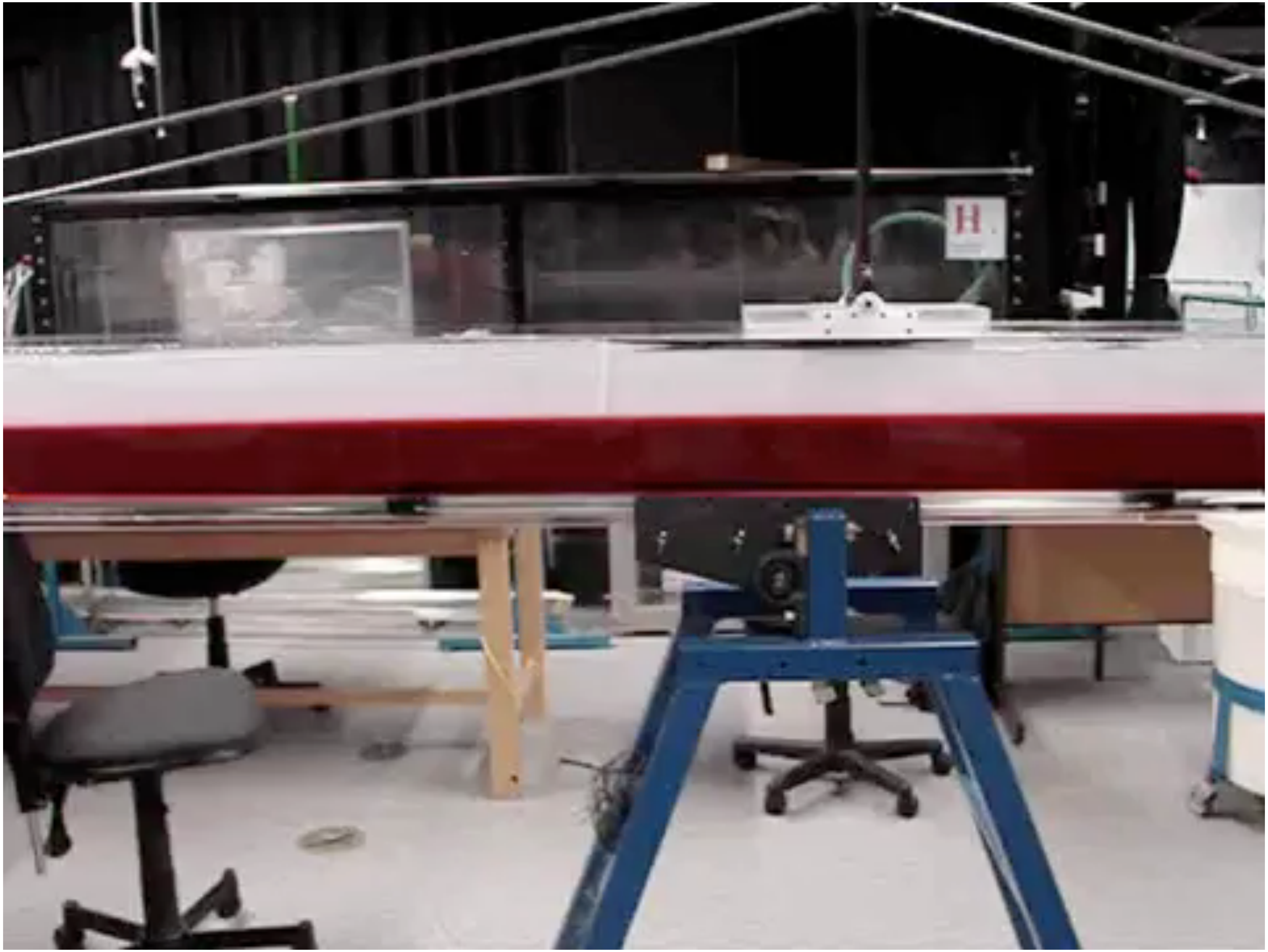
Edgington-Mitchell et al. 2017





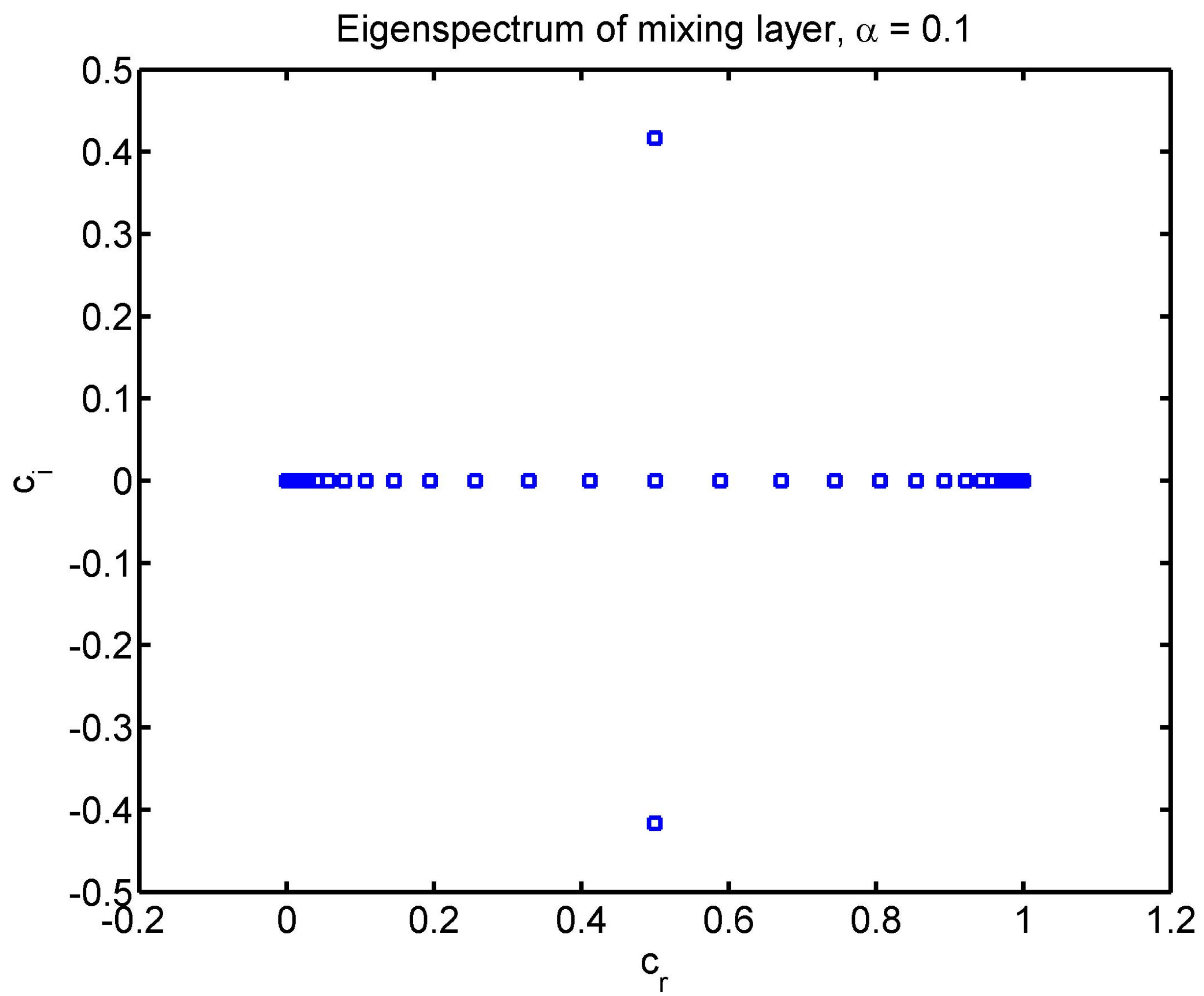


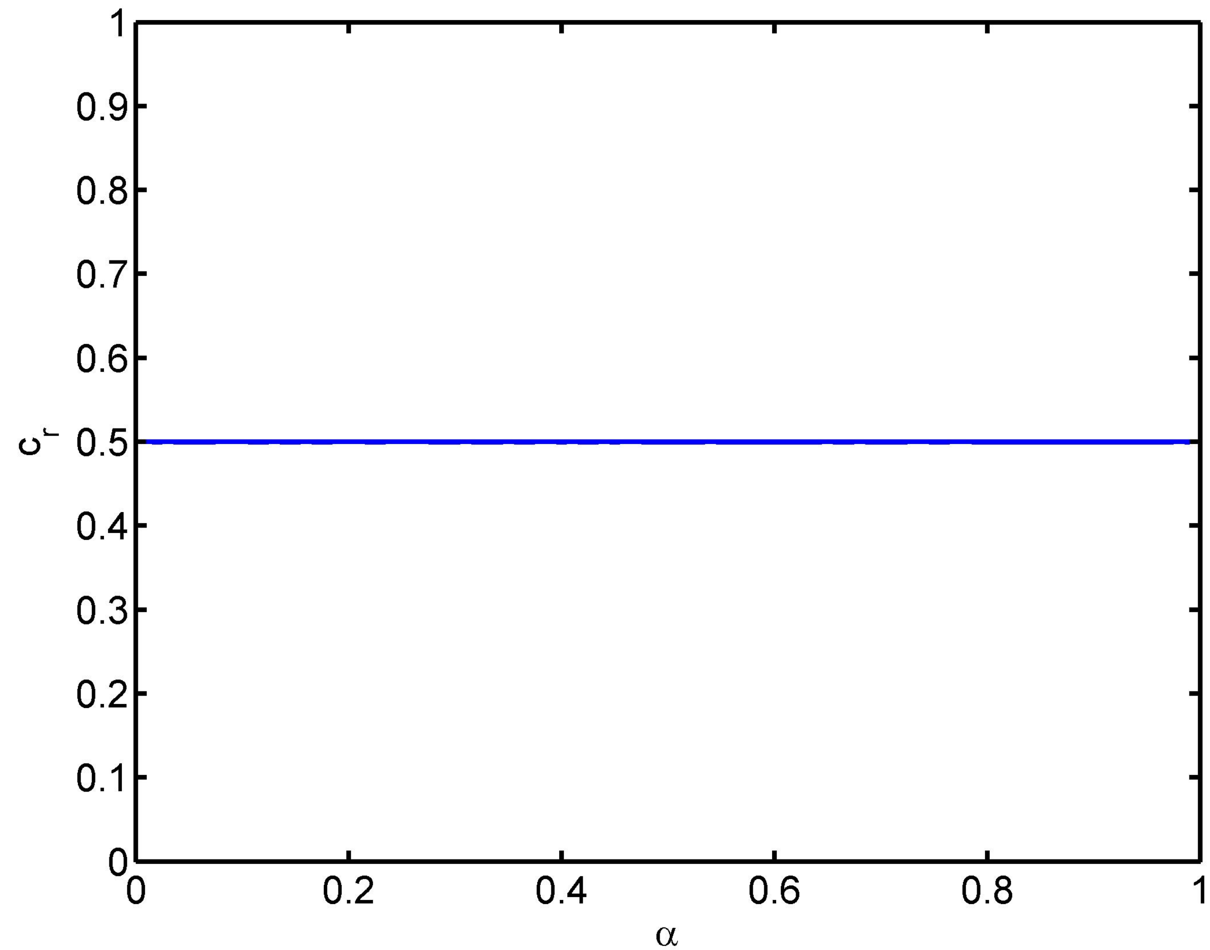
3. Inviscid temporal stability of the mixing layer



Expected results - phase speed

$$\tilde{v}(x, y, t) = \hat{v}\mathrm{e}^{i\alpha(x-ct)}$$
$$\omega_i = \alpha c_i$$



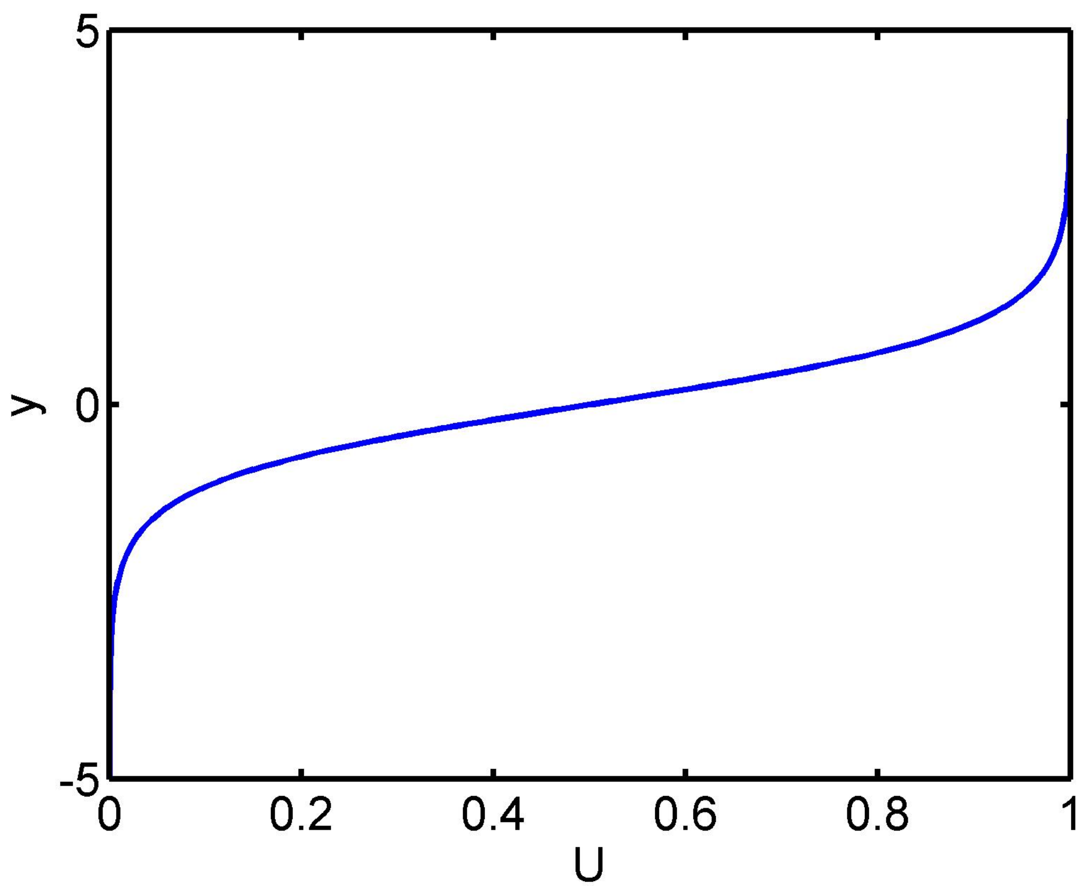


3. Inviscid temporal stability of the mixing layer

Expected results - phase speed

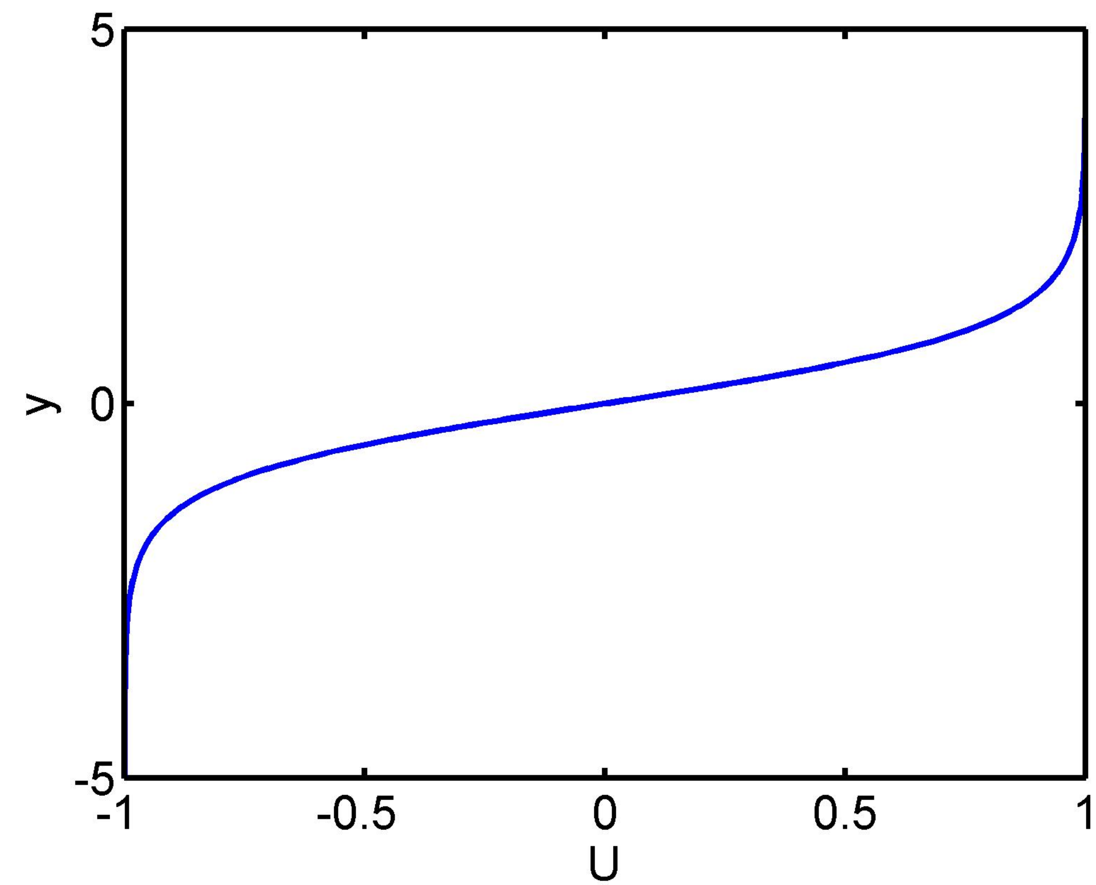
$$\tilde{v}(x,y,t) = \hat{v}\mathrm{e}^{i\alpha(x-ct)}$$
$$\omega_i = \alpha c_i$$

Michalke 1964, U = ½(1+tanh(y))



Result: c_r (K-H) = 0.5

U = tanh(y)

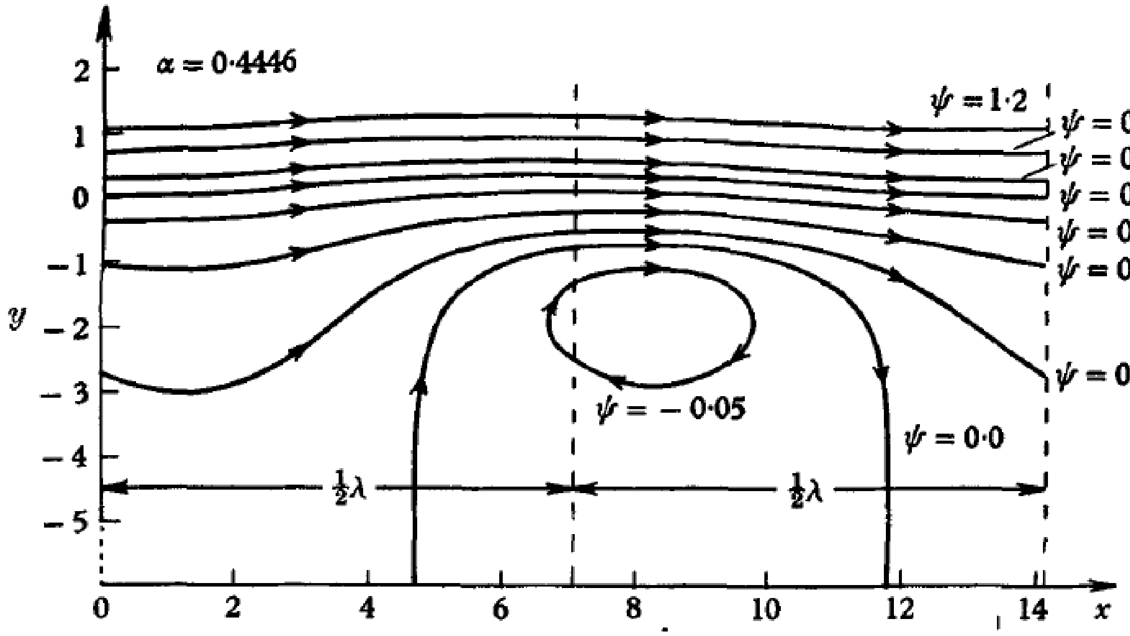


Result: c_r (K-H) = 0

Expected results - flow structure

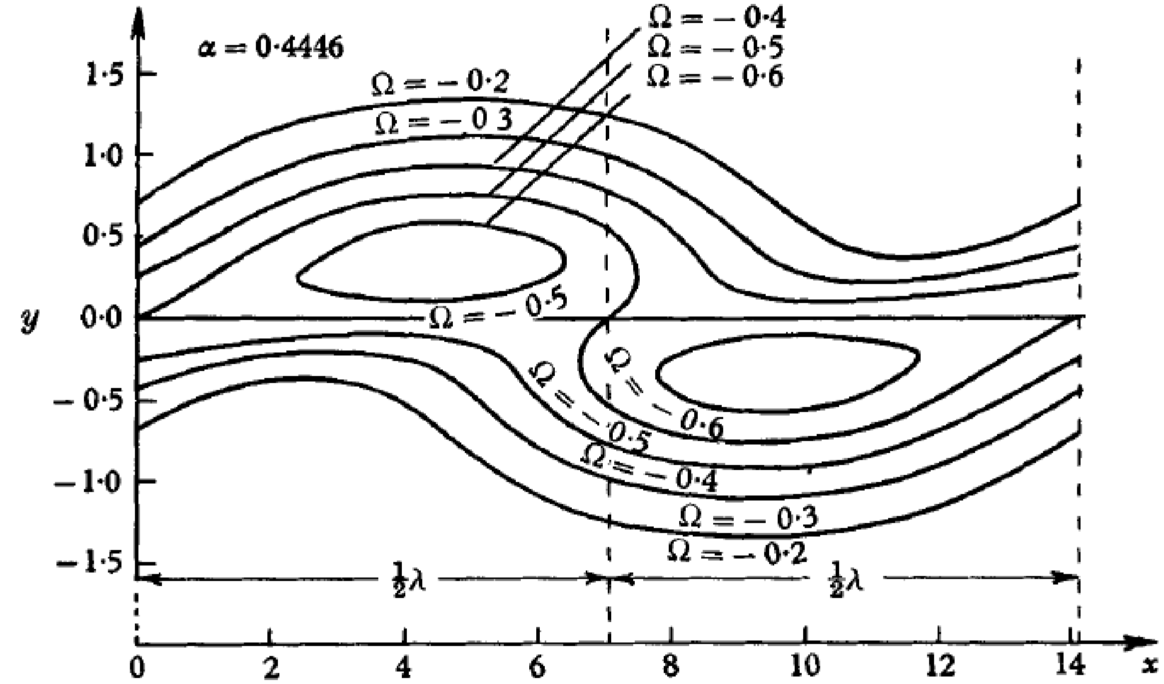
$$\tilde{v}(x,y,t) = \hat{v}\mathrm{e}^{i\alpha(x-ct)}$$
$$\omega_i = \alpha c_i$$

Eigenfunctions allow study of structure of growing disturbances



Streamlines

Michalke 1964

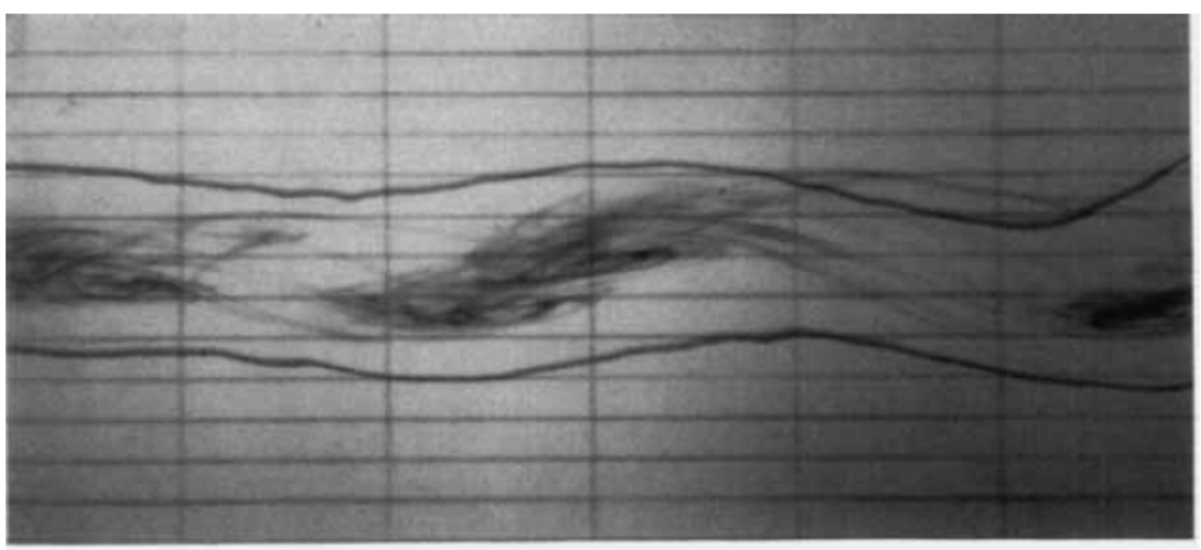


Vorticity

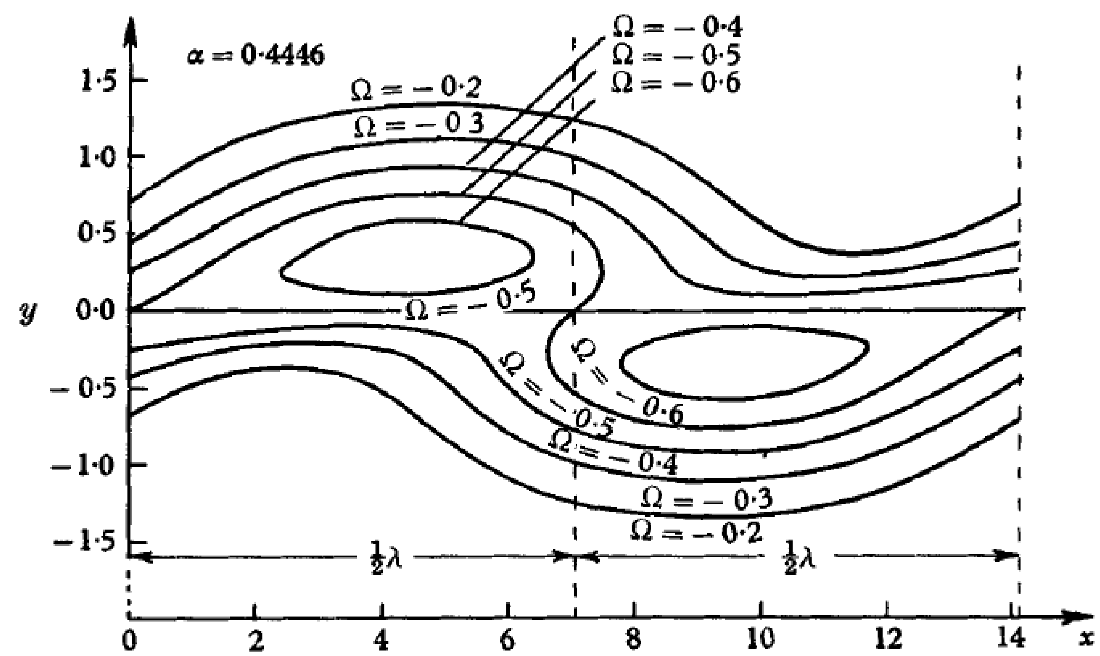
Expected results - flow structure

$$\tilde{v}(x,y,t) = \hat{v}\mathrm{e}^{i\alpha(x-ct)}$$
$$\omega_i = \alpha c_i$$

Eigenfunctions allow study of structure of growing disturbances



(Winant & Browand 1974)

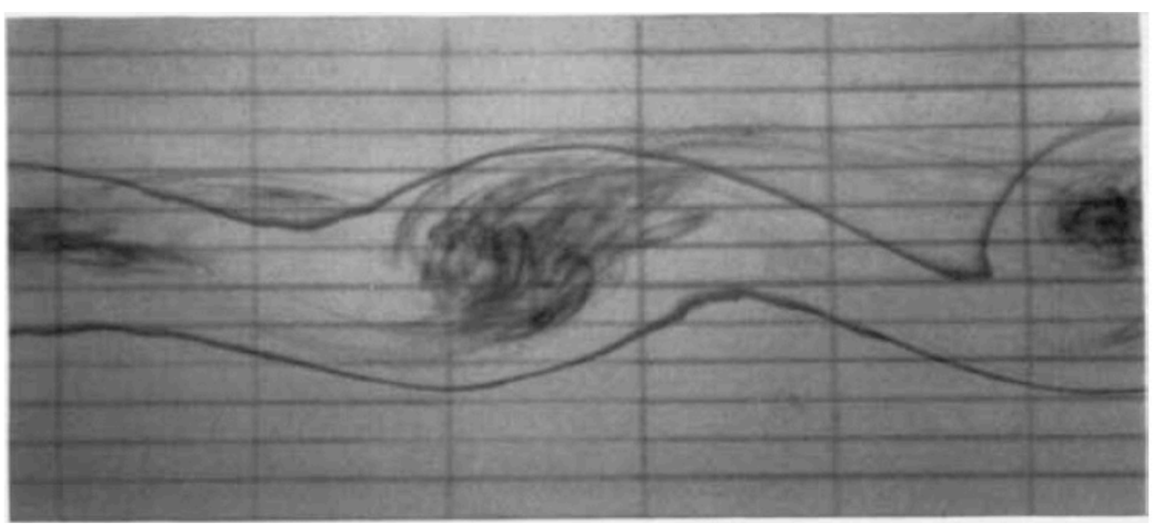


Vorticity

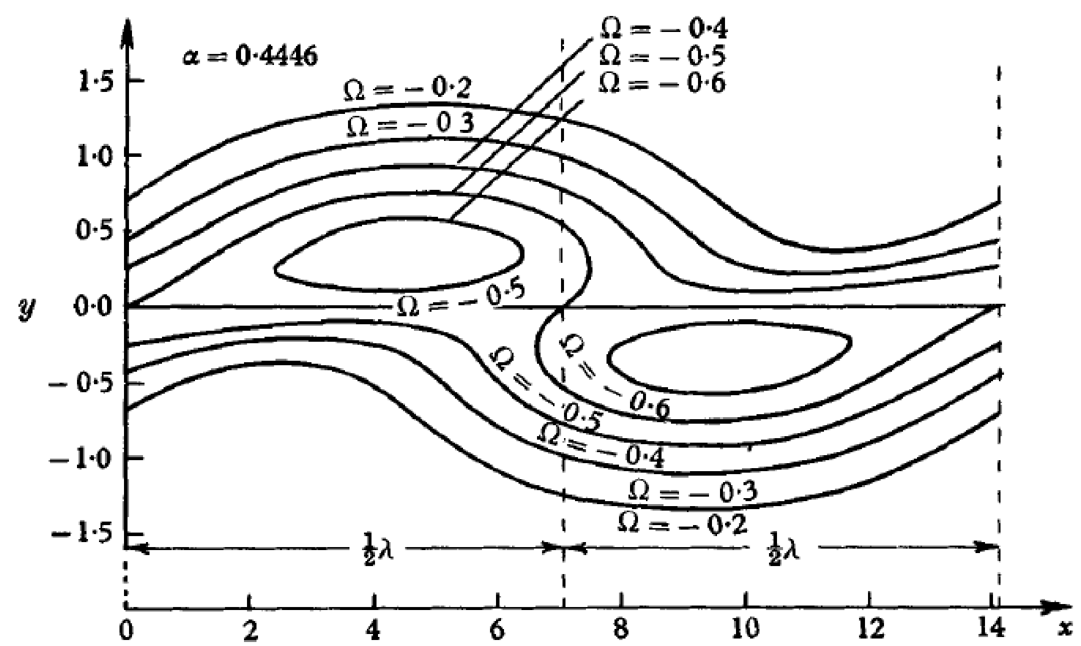
Expected results - flow structure

$$\tilde{v}(x,y,t) = \hat{v}\mathrm{e}^{i\alpha(x-ct)}$$
$$\omega_i = \alpha c_i$$

Eigenfunctions allow study of structure of growing disturbances



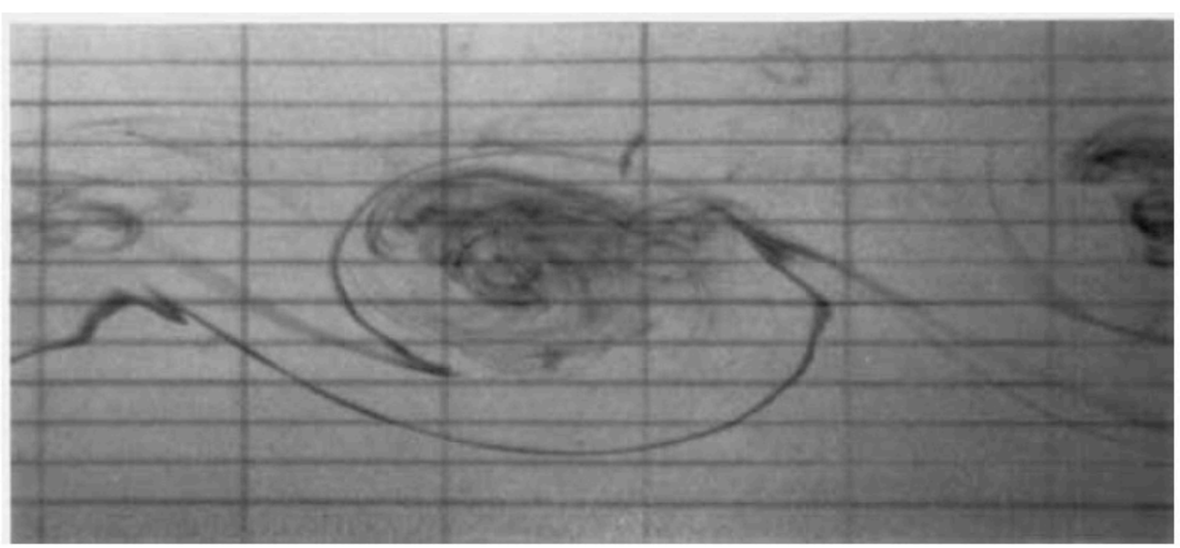
(Winant & Browand 1974)



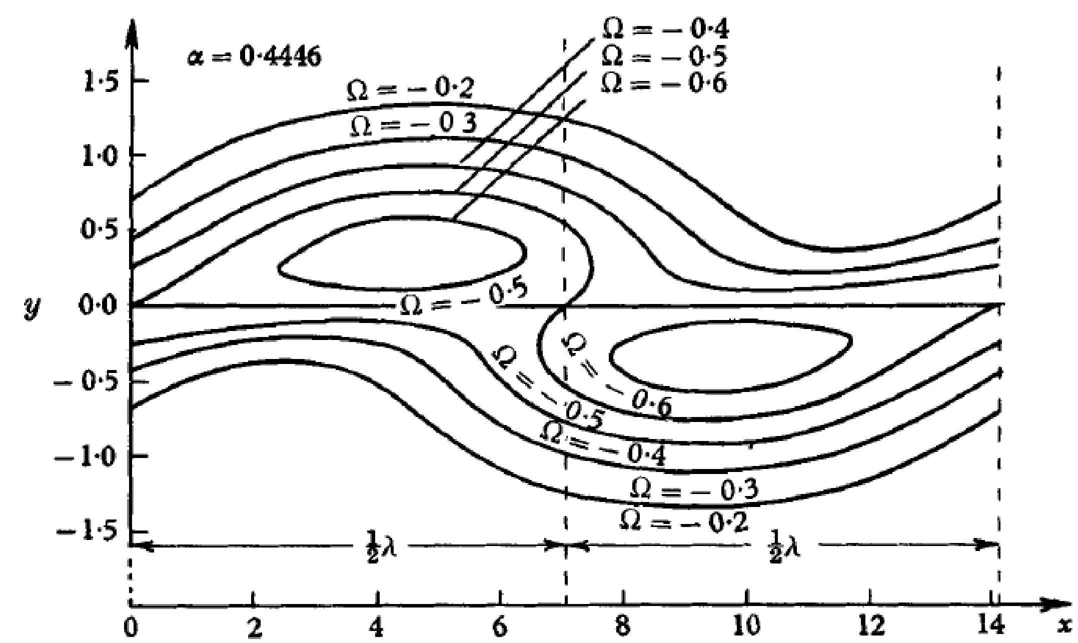
Expected results - flow structure

$$\tilde{v}(x,y,t) = \hat{v}\mathrm{e}^{i\alpha(x-ct)}$$
$$\omega_i = \alpha c_i$$

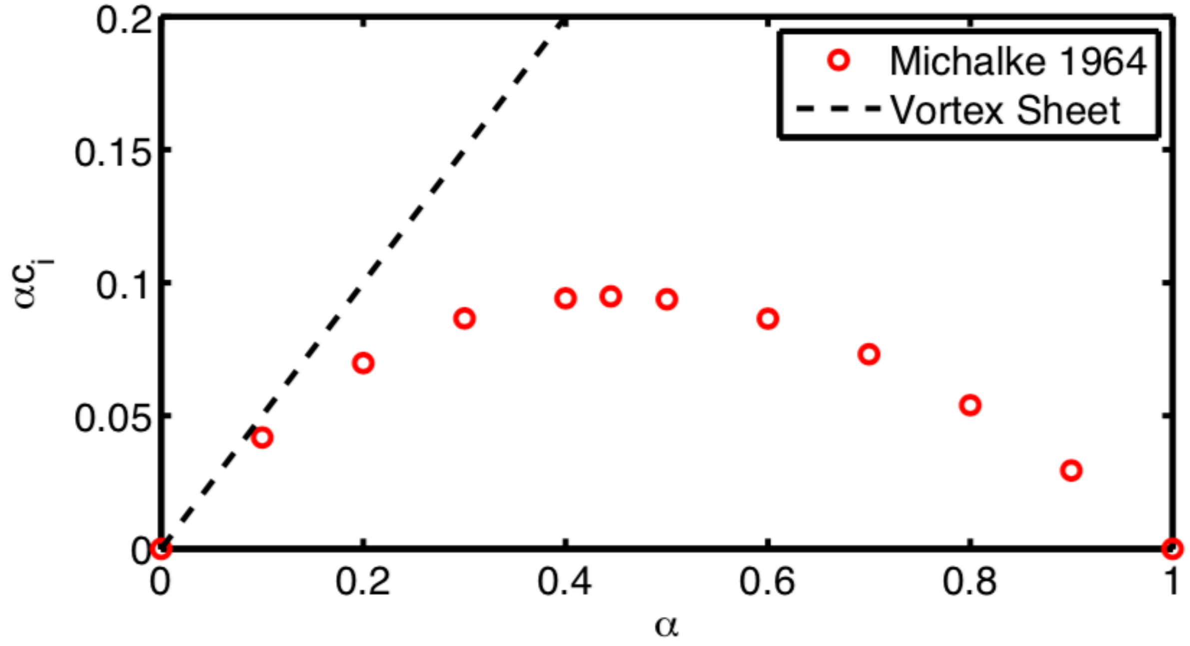
Eigenfunctions allow study of structure of growing disturbances

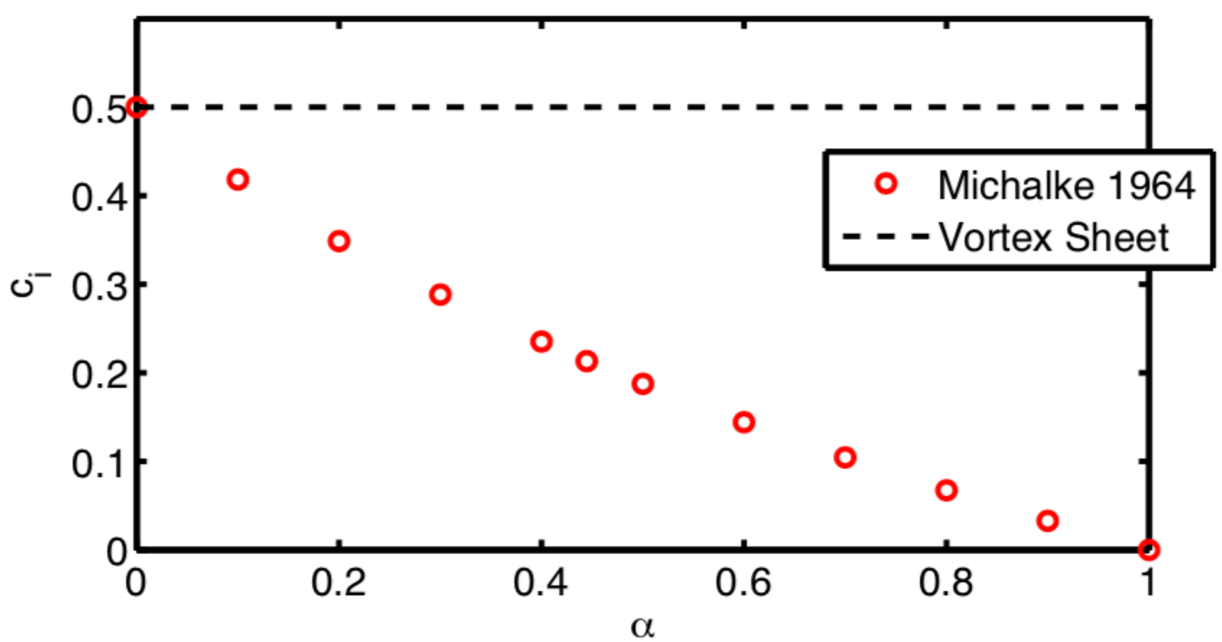


(Winant & Browand 1974)



Comparison with vortex-sheet model





Discussion

Rayleigh equation neglects viscous effects:

- no information on critical Reynolds number
- asymptotic behaviour for Re->infinity
- predicts initial stage of transition for high Re

Rayleigh equation becomes singular for c=U:

- critical layer
- special care necessary to treat singularity,
- easiest option is to account for viscosity

Orr-Sommerfeld equation

$$(U-c)\left(\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2}-\alpha^2\mathbf{v}(y)\right)-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\mathbf{v}(y)=\frac{1}{i\alpha\mathrm{Re}}\left(\frac{\mathrm{d}^4\mathbf{v}(y)}{\mathrm{d}y^4}-2\alpha^2\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2}+\alpha^4\mathbf{v}(y)\right)$$

Discussion

In unsteady, turbulent flows, clear understanding is exception rather than rule:

- non-linear system of PDEs, 4 dimensions, 5 dependent variables…

Linear stability theory allows identification of a well-defined flow feature:

- an instability wave, appropriate for initial stages of transition.

Coherent structures in turbulent flows can also be modelled as instability waves.

Eigenfunctions from stability analysis form a complete basis, albeit non-orthogonal:

- it is possible to project flow data onto the eigenfunctions to obtain amplitudes of each of the modes (Rodriguez et al. 2013, 2015)

Problem is much cheaper to solve numerically than DNS, LES:

- potential for reduced-order modelling and control

Résumé

Matrix representation of linearised equations

- Finite difference versus pseudo spectral

Linearised flow equations formulated as a generalised eigenvalue problem

Rayleigh equation - calculation of inviscid instability

- solution of flow equations for mixing-layer transition
- instability waves non-dispersive