INSTITUT PPRIME CNRS–UPR–3346 *•* UNIVERSITE DE POITIERS ´ *•* ENSMA

> DÉPARTEMENT D2 – FLUIDES THERMIQUE ET COMBUSTION

December 11, 2012 **An introduction to hydrodynamic stability**

Cambridge University Department of Engineering **Lecture 2: The governing equations for fluid instability**

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It is my pleasure to provide this appraisal of Dr. Agarwal's contribution to the **peter.jordan@univ-poitiers.fr**

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1. A quick recap. of lecture 1

2. Rayleigh and Orr-Sommerfeld equations

3. The Squire transformation

1. Recap. of lecture 1

1 Linearised dynamics

Determining stability via consideration of ENERGY Consider a first order linear differential equation describing the time evolution of **Setermining stability via consideration of Energy**

Determining stability via consideration of LINEAR DYNAMICS Consider a first order linear differential experience of the time and the time the time evolution of the time time the time evolution of the time evolution of the time evolution of the time of the time evolution of the tim socomming otability via conoidoration of Envernit B Fits nsi = *L*(*u*) (1)

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Connection with fluid mechanics?

Kelvin-Helmholtz instability \mathbf{F}

Ξ Ξ

1*,*2 1*,*2

Ĩ.

1*,*2

@*z*² = 0 (15)

²

u

u

or

لي المسلمات
Potential flow assumed above and below the vortex sheet: Laplace's equation If the real part of either *^s*¹ t hace's equation
discrete
also discrete
also discrete
 t gives, respectively. c respectively. 1*,*² = *± i ow the vortex sheet: Laplace's equation* $\overline{\text{th}}$ \cdot orte brtex sneet: Lapiace's equation ⌘0*/* = (*s* + *iU*)

$$
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0
$$

i

@*x*

i^x .
*** \mathbf{r} er
2020 **Disturbance Ansatz: velocity potential with normal modes:**
 Disturbance Ansatz: velocity potential with normal modes: z: velocity potential with normal modes:

$$
\phi(x, z, t) = Ux + f(z)e^{st + i\kappa x}
$$

Leads to ODE for transverse structure: الاه.
الم

$$
\varphi(x, z, v) = c \, x + f(z) \mathbf{1}
$$
\nis also to ODE for transverse structure:

\n
$$
-\kappa^2 f(z) + \frac{d^2 f(z)}{dz^2} = 0
$$
\n
$$
f(z) = A_1 e^{-\kappa z} + A_2 e^{\kappa z}
$$

@*^z* @*^w*

@*x* :

2 **General solution:**

$$
dz^{2}
$$

eneral solution: $f(z) = A_1 e^{-\kappa z} + A_2 e^{\kappa z}$
 $f(z) = B_1 e^{-\kappa z} + B_2 e^{\kappa z}$

 \overline{c} **Boundary and interface matching conditions:**

boundary and interface matching conditions:

\n
$$
\eta(x,t) = \eta_0 e^{\frac{1}{2}\kappa U t + i\kappa (x - \frac{1}{2}U t)}
$$
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2. Rayleigh & Orr-Sommerfeld equations

The general approach for stabilty analysis can be understood by considering a simplified flow configuration:

- Parallel, 2D, shear flow,

which has surprisingly widespread applicability

General approach:

- **1. Equations of motion (mass and momentum conservation)**
- **2. Non-dimensionalisation**
- **3. Identification of BASE-FLOW (steady laminar solution)**
- **4. Decomposition of dependent variables into STEADY & FLUCTUATING quantities**
- **5. Substitution into equations of motion**
- **6. LINEARISATION (subtract base-flow equations; remove non-linear terms)**
- **7. Reduce linearised equations to some compact form (often a single equation)**
- **8. Express dependent variables in terms of NORMAL MODES**
- **9. Introduction into linearised equation:**

- PDE system becomes a single ODE, but with too many unknowns 10. Specify a value for one of the unknowns (wavenumber for instance), solve for others: generally an EIGENVALUE PROBLEM

1. Equations of motion, in 2D, for incompressible, isentropic, flow: in 2D, for incomp $\overline{\text{ressible, isentropic, flow:}}$

Mass conservation: Mass conservation:

<u>Example</u>

$$
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
$$

⇢ $$ **Momentum conservation:**

$$
\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + \frac{\partial p}{\partial x} = \mu \nabla^2 u
$$

$$
\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) + \frac{\partial p}{\partial y} = \mu \nabla^2 v
$$

Fluid motions constrained by:

- **Mass conservation**
- **Momentum conservation**
- **Boundary conditions**

1. Equations of motion, in 2D, for incompressible, isentropic, flow: in 2D, for incomp $\overline{\text{ressible, isentropic, flow:}}$

Mass conservation: Mass conservation:

<u>Example</u>

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$$

$$
\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) + \frac{\partial p}{\partial y} = \mu \nabla^2 v
$$

2. Non-dimensionalisation 2. Non-dimensionalisation

$$
u = \frac{u^+}{U_c}
$$

$$
v = \frac{v^+}{U_c}
$$

$$
p = \frac{p^+}{\rho U_c^2}
$$

$$
x = \frac{x^{+}}{L}
$$

$$
y = \frac{y^{+}}{L}
$$

$$
t = \frac{t^{+}U_{c}}{L}
$$

p =

⇢*U*²

1. Equations of motion, in 2
2. Non-dimensionalisation 1. Equations of motion, in 2D, for incompressible, isentropic, flow: *y*+ *L* (62) **L** (6

Mass conservation:

$$
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
$$

^L (61)

⇢ $$ **Momentum conservation:** @*^t* ⁺ *^u*

$$
\textrm{Re}=\tfrac{\rho L U_c}{\mu}
$$

$$
\text{Re} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \text{Re}^{-1} \nabla^2 u
$$
\n
$$
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \text{Re}^{-1} \nabla^2 v
$$

@*y*

3. Identification of BASE-FLOW

- Parallel and 2D (if flow changes slowly in some direction a locally parallel **approximation is often adequate)** approximation is often adequate)

 $U(y)$

4. Decomposition into STEADY & FLUCTUATING quantities

$$
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
$$
\n
$$
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \text{Re}^{-1} \nabla^2 u
$$
\n
$$
u(x, y, t) = U(y) + \tilde{u}(x, y, t)
$$
\n
$$
v(x, y, t) = \tilde{v}(x, y, t)
$$
\n
$$
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \text{Re}^{-1} \nabla^2 v
$$

$$
u(x, y, t) = U(y) + \tilde{u}(x, y, t)
$$

\n
$$
v(x, y, t) = \tilde{v}(x, y, t)
$$

\n
$$
p(x, y, t) = P(x) + \tilde{p}(x, y, t)
$$

\n
$$
v
$$

5. Substitute into equations of motion & subtract base-flow equations

$$
\overline{\mathscr{S}}
$$

$$
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0
$$

$$
\frac{\partial \tilde{u}}{\partial t} + U \frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y} \tilde{v} + \frac{\partial \tilde{p}}{\partial x} + \left(\tilde{u} \frac{\partial \tilde{u}}{\partial x} + \tilde{v} \frac{\partial \tilde{u}}{\partial y}\right) = \mathrm{Re}^{-1} \nabla^2 \tilde{u}
$$

$$
\frac{\partial \tilde{v}}{\partial t} + U \frac{\partial \tilde{v}}{\partial x} + U \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} + \left(\tilde{u} \frac{\partial \tilde{v}}{\partial x} + \tilde{v} \frac{\partial \tilde{v}}{\partial y}\right) = \mathrm{Re}^{-1} \nabla^2 \tilde{v}
$$

 $\partial \tilde{v}$

6. LINEARISATION

6. LINEARISATION

 \mathbb{R}^n

r2

$$
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0
$$

$$
\frac{\partial \tilde{u}}{\partial t} + U \frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y} \tilde{v} + \frac{\partial \tilde{p}}{\partial x} = \mathrm{Re}^{-1} \nabla^2 \tilde{u}
$$

$$
\frac{\partial \tilde{v}}{\partial t} + U \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} = \mathrm{Re}^{-1} \nabla^2 \tilde{v}
$$

⇣

@*v*˜

@*v*˜

@*v*˜

@*v*˜

⌘

⌘

7. Reduction of linearised system to more COMPACT FORM The Puppe of the initial stages of the transition process. The linear system is of the initial stages of the transition process. The linear

$$
\begin{array}{|c|c|}\n\hline\n\frac{\partial \tilde{u}}{\partial t} + U \frac{\partial \tilde{u}}{\partial x} + \frac{\partial U}{\partial y} & = 0 \\
\frac{\partial \tilde{u}}{\partial t} + U \frac{\partial \tilde{u}}{\partial x} + \frac{\partial U}{\partial y} \tilde{v} + \frac{\partial \tilde{p}}{\partial x} & = \text{Re}^{-1} \nabla^2 \tilde{u} \\
\frac{\partial \tilde{v}}{\partial t} + U \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} & = \text{Re}^{-1} \nabla^2 \tilde{v}\n\end{array}
$$

From this system of equations a number of equivalent systems can be derived

tions.

From this system of equations a number of equivalent systems can be derived

tivation of the non-linear terms, the non-linearised system is sufficient for description $\mathcal{L}_\mathcal{A}$

VELOCITY disturbance equation where one or two differential equations capture, describe, constrain & explain the **VELOCITY disturbance equation** equations with two unknowns: mass continuity and a second-order PDE for *u* and *v*. Differentiation the latter write write write \mathbf{u} and eliminating \mathbf{u} using the continuity equation \mathbf{u}

- **Divergence of mom. eqs.**
- **Eliminate divergence-free terms**
- **Take Laplacian of v-mom. eq.**
- **Eliminate pressure**

$$
\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 \tilde{v} - \frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\frac{\partial \tilde{v}}{\partial x} = \mathrm{Re}^{-1}\nabla^4 \tilde{v}
$$

This single fourth-order equation for *v* can be solved given boundary condi-

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7. Reduction of linearised system to more COMPACT FORM The Puppe of the initial stages of the transition process. The linear system is of the initial stages of the transition process. The linear Taking the current of the two momentum equations and subtituting in the continuum equations are continued in the continuous continuous continuous continuous continuous continuous continuous continuous continuous continuous

2.1 Velocity disturbance equation

$$
\begin{array}{|c|c|}\n\hline\n\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} & = 0 \\
\frac{\partial \tilde{u}}{\partial t} + U \frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y} \tilde{v} + \frac{\partial \tilde{p}}{\partial x} & = \mathrm{Re}^{-1} \nabla^2 \tilde{u} \\
\frac{\partial \tilde{v}}{\partial t} + U \frac{\partial \tilde{v}}{\partial x} & + \frac{\partial \tilde{p}}{\partial y} & = \mathrm{Re}^{-1} \nabla^2 \tilde{v} \\
\hline\n\text{Fluid} \\
\hline\n\end{array}
$$

From this system of equations a number of equivalent systems can be derived

tivation of the non-linear terms, the non-linearised system is sufficient for description $\mathcal{L}_\mathcal{A}$

STREAMFUNGTION disturbance equation where one or two differential equations capture, describe, constrain & explain the **STREAMFUNCTION disturbance equation**

From this system of equations a number of equivalent systems can be derived

- Substitute streamfunction - Curl of momentum equations (which automatically satisfies continuity equation)

$$
\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla^2 \Psi - \frac{\mathrm{d}^2 U}{\mathrm{d} y^2} \frac{\partial \Psi}{\partial x} = \mathrm{Re}^{-1} \nabla^4 \Psi
$$

INSTITUT PPRIME 2.3 Pressure disturbance equation \mathcal{L} pressure disturbance equation \mathcal{L}

7. Reduction of linearised system to more COMPACT FORM The Puppe of the initial stages of the transition process. The linear system is of the initial stages of the transition process. The linear sed sys⁻

$$
\begin{array}{|c|c|} \hline & \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} & = 0 \\ \hline \frac{\partial \tilde{u}}{\partial t} + U \frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y} \tilde{v} + \frac{\partial \tilde{p}}{\partial x} & = \mathrm{Re}^{-1} \nabla^2 \tilde{u} \\ \hline \frac{\partial \tilde{v}}{\partial t} + U \frac{\partial \tilde{v}}{\partial x} & + \frac{\partial \tilde{p}}{\partial y} & = \mathrm{Re}^{-1} \nabla^2 \tilde{v} \\ \hline \hline \end{array}
$$

From this system of equations a number of equivalent systems can be derived

sure.

From this system of equations a number of equivalent systems can be derived

tivation of the non-linear terms, the non-linearised system is sufficient for description $\mathcal{L}_\mathcal{A}$

PRESSURE disturbance equation. where one or two differential equations capture, describe, constrain & explain the **PRESSURE disturbance equation**

- **Differentiate u- and v- mom. equations w.r.t. x & y** 2.3 Pressure disturbance equation
- **Add them & simplify using continuity**
- **Material derivative of result**
- **Eliminate dv/dt using mom. eq.**

$$
\begin{vmatrix}\n\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\n\end{vmatrix}\nabla^2 \tilde{p} - 2 \frac{dU}{dy} \frac{\partial \tilde{p}}{\partial x \partial y} = -2 \frac{dU}{dy} \text{Re}^{-1} \nabla^2 \frac{\partial \tilde{v}}{\partial x}
$$
\nfrom, eq.

INSTITUT PPRIME What all of these simplifications are telling us in the dynamics is that the dynamics is that the dynamics is s
In the dynamics is simplified us is simplified us in the dynamics is simplified us in the dynamics is simplifi

7. Reduction of linearised system to more COMPACT FORM The Puppe of the initial stages of the transition process. The linear system is of the initial stages of the transition process. The linear

$$
\begin{array}{|c|c|}\n\hline\n\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} & = 0 \\
\frac{\partial \tilde{u}}{\partial t} + U \frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y} \tilde{v} + \frac{\partial \tilde{p}}{\partial x} & = \mathrm{Re}^{-1} \nabla^2 \tilde{u} \\
\frac{\partial \tilde{v}}{\partial t} + U \frac{\partial \tilde{v}}{\partial x} & + \frac{\partial \tilde{p}}{\partial y} & = \mathrm{Re}^{-1} \nabla^2 \tilde{v}\n\end{array}
$$

From this system of equations a number of equivalent systems can be derived

From this system of equations a number of equivalent systems can be derived

tivation of the non-linear terms, the non-linearised system is sufficient for description $\mathcal{L}_\mathcal{A}$

SORIGITY disturbance equation where one or two differential equations capture, describe, constrain & explain the **VORTICITY disturbance equation** $\boldsymbol{\mathrm{equation}}$ can be obtained by taking the momentum equation by taking the momentum equal of the momentum equal o

7 7 **- Curl of momentum equations which automatically eliminates pressure**

$$
\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \tilde{\omega}_z - \frac{\mathrm{d}^2 U}{\mathrm{d} y^2} \tilde{v} = \mathrm{Re}^{-1} \nabla^2 \tilde{\omega}_z \right)
$$

Equation is analogous to the equation for heat conduction-diffusion in the pres-

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2

is the complex phase velocity of the wave; the frequency is ! = ↵*c*. The non-

8. Express dependent variables as NORMAL MODES \overline{a} real disturbance is a real disturbance is \overline{a} 8. Express dependent variables as NORMAL MODES

temperature in a conduction-diffusion-diffusion-diffusion-diffusion-diffusion-diffusion-diffusion-diffusion-diffusion-diffusion-diffusion-diffusion-diffusion-diffusion-diffusion-diffusion-diffusion-diffusion-diffusion-dif

- **Decompose the** *x* **and** *t* **directions into Fourier modes** - Decompose the x and t directions into Fourier modes
	- Streamwise spatial structure expanded as spectrum of spatial modes
(sines & cosines) characterised by their WAVENHMRERS **(sines & cosines) characterised by their WAVENUMBERS** whose imaginary part gives the wave undergoes tells us that the wave undergoes exponential temporal temporal t
Whose interpretations in the wave undergoes exponential temporal temporal temporal temporal temporal temporal

$$
\alpha\,=\,\alpha_r\,+\,i\alpha_i
$$

- Temporal structure (*t*-direction) expanded as spectrum of temporal modes
(sines & cosines) characterised by their FRFOUFNCIFS **(sines & cosines) characterised by their FREQUENCIES** is the complex phase velocity of the wave; the frequency is ! = ↵*c*. The non- \mathbf{r} or temporar modes.
S In the general case, we have the amplitudes of the disturbances proportional - Temporal structure (t-direction) expanded as spectrum of temporal modes face a confest of the actual be well the system. The controlled to \sim

$$
\omega=\omega_r+i\omega_i
$$

- Perturbations can be treated as a superposition of waves travelling at speed *c***:** *<u>reated as a superposition of waves travelling at*</u> The advantage of using complex quantities is as follows. An oscillation has a $\left(\frac{a}{b}\right)^n$ *^u*˜(*x, y, t*) = ¹

$$
e^{i\alpha(x-ct)} \qquad c = c_r + ic_i
$$

8. Express dependent variables as NORMAL MODES *p* ˆ = p(*y*)e⁴*i*(*x*(0*.*7+0*.*2*i*)*t*) (86)

Complex frequency is

\n
$$
\omega = \omega_r + i\omega_i
$$
\n
$$
\omega_r = \alpha_r c_r - \alpha_i c_i
$$
\n
$$
\omega_i = \alpha_r c_i + \alpha_i c_r
$$

General disturbance
$$
e^{i\alpha(x-ct)} = e^{i(\alpha_r + i\alpha_i)x}e^{-i(\alpha_r + i\alpha_i)(c_r + ic_i)t}
$$

In the general case, we have that case, we have the amplitudes of the disturbances of the disturbances proportional case of the disturbances proportional case of the disturbances proportional case of the disturbances prop Show that

$$
= \mathrm{e}^{i(\alpha_rx-\omega_rt)} \mathrm{e}^{-\alpha_ix+\omega_it}
$$

8. Express dependent variables as NORMAL MODES *p* ˆ = p(*y*)e⁴*i*(*x*(0*.*7+0*.*2*i*)*t*) (86)

Complex frequency is

\n
$$
\omega = \omega_r + i\omega_i
$$
\n
$$
\omega_r = \alpha_r c_r - \alpha_i c_i
$$
\n
$$
\omega_i = \alpha_r c_i + \alpha_i c_r
$$

Show that

General disturbance
\n
$$
e^{i\alpha(x-ct)} = e^{i(\alpha_r + i\alpha_i)x}e^{-i(\alpha_r + i\alpha_i)(c_r + ic_i)t}
$$
\n
$$
e^{i(\alpha_r + i\alpha_i)x}e^{-i(\alpha_r c_r + ic_i\alpha_r + ic_r\alpha_i + i^2 c_i\alpha_i)t}
$$
\n
$$
e^{i(\alpha_r + i\alpha_i)x}e^{-i(\alpha_r c_r + ic_i\alpha_r + ic_r\alpha_i - c_i\alpha_i)t}
$$
\n**Show that**
\n
$$
e^{i(\alpha_r + i\alpha_i)x}e^{-i(\alpha_r c_r - c_i\alpha_i)t - i^2(c_i\alpha_r + c_r\alpha_i)t}
$$
\n
$$
e^{i(\alpha_r + i\alpha_i)x}e^{-i\omega_r t + \omega_i t}
$$
\n
$$
e^{i\alpha_r x - \alpha_i x}e^{-i\omega_r t + \omega_i t}
$$
\n
$$
= e^{i(\alpha_r x - \omega_r t)}e^{-\alpha_i x + \omega_i t}
$$

8. Express dependent variables as NORMAL MODES \blacksquare *MODES* **d.** Expless dependent variables as NOTHVIAL IVIODES

General disturbance proportional to $\frac{1}{2}$

e*ⁱ*(↵*r*+*i*↵*i*)*^x*e*i*(↵*rcr*+*ici*↵*r*+*icr*↵*i*+*i*2*ci*↵*i*)*^t* (93)

8. Express dependent variables as NORMAL MODES e*ⁱ*(↵*r*+*i*↵*i*)*^x*e*i*(↵*rcr*+*ici*↵*r*+*icr*↵*ici*↵*i*)*^t* (94) e*ⁱ*(↵*r*+*i*↵*i*)*^x*e*i*(↵*rcrci*↵*i*)*ti*2(*ci*↵*r*+*cr*↵*i*)*^t* (95) e*ⁱ*(↵*r*+*i*↵*i*)*^x*e*i*(↵*rcr*+*ici*↵*r*+*icr*↵*i*+*i*2*ci*↵*i*)*^t* (93) $\mathbf{r} = \mathbf{r} \cdot \mathbf{r$

General disturbance proportional to
 ei e*ⁱ*(↵*r*+*i*↵*i*)*^x*e*i*(↵*rcrci*↵*i*)*ti*2(*ci*↵*r*+*cr*↵*i*)*^t* (95) e*ⁱ*(↵*r*+*i*↵*i*)*^x*e*i*!*rt*+!*i^t* (96)

time implies that the frequency is real: ↵ then becomes a complex eigenvalue,

$$
e^{i\alpha(x-ct)} = e^{i(\alpha_rx-\omega_rt)}e^{-\alpha_ix+\omega_it}
$$

problem. The temporal stability problem is when ↵*ⁱ* = 0, ↵*^r* is specified and !

Substitution into the governing linearised equations reduces the system to an

e*ⁱ*(↵*r*+*i*↵*i*)*^x*e*i*(↵*rcr*+*ici*↵*r*+*icr*↵*i*+*i*2*ci*↵*i*)*^t* (93)

9. Substitute into linearised equations of motion \blacksquare

$$
\frac{\partial \tilde{u}}{\partial t} + U \frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y} \tilde{v} + \frac{\partial \tilde{p}}{\partial x} = \mathbf{R} \mathbf{e}^{-1} \nabla^2 \tilde{u}
$$
\n
$$
\frac{\partial \tilde{v}}{\partial t} + U \frac{\partial \tilde{v}}{\partial x} + U \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} = \mathbf{R} \mathbf{e}^{-1} \nabla^2 \tilde{u}
$$
\n
$$
\frac{\partial \tilde{v}}{\partial t} + U \frac{\partial \tilde{v}}{\partial x} + U \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} = \mathbf{R} \mathbf{e}^{-1} \nabla^2 \tilde{v}
$$
\n
$$
\frac{\partial \tilde{v}}{\partial t} + U \frac{\partial \tilde{v}}{\partial x} + U \frac{\partial \tilde{v}}{\partial y} = \mathbf{R} \mathbf{e}^{-1} \nabla^2 \tilde{v}
$$
\n
$$
\frac{\partial \tilde{v}}{\partial t} + U \frac{\partial \tilde{v}}{\partial x} + U \frac{\partial \tilde{v}}{\partial y} = \mathbf{R} \mathbf{e}^{-1} \nabla^2 \tilde{v}
$$

or two differential equations capture, the normal modes of the normal modes of the normal modes, the real disturbance of the normal modes, the normal modes will be solved in the normal modes, the real disturbance of the no same dynamics. We will not derive them in full (try this as an exercise). can equality pressure the above. **Orr-Sommerfeld equation (1907-1908)** Our Semmerfold equation (1007 1000) to the Orrest Communication: Communication: Communication: Communication: Communication: Communication: Commun
The Orrest Communication: Communication: Communication: Communication: Communication: Communication: Communica

$$
(U-c)\left(\frac{d^2\mathbf{v}(y)}{dy^2} - \alpha^2\mathbf{v}(y)\right) - \frac{d^2U}{dy^2}\mathbf{v}(y) = \frac{1}{i\alpha \text{Re}}\left(\frac{d^4\mathbf{v}(y)}{dy^4} - 2\alpha^2\frac{d^2\mathbf{v}(y)}{dy^2} + \alpha^4\mathbf{v}(y)\right)
$$

I N S T I T U T P P R I M E

facilitated by the linearity of the system, the system, the system, the above PDEs can be reduced to ODEs can
The above PDEs can be reduced to ODEs can

9. Substitute into linearised equations of motion s of motion

$$
\frac{\partial \tilde{u}}{\partial t} + U \frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y} \tilde{v} + \frac{\partial \tilde{p}}{\partial x} = \mathbf{R} e^{-1} \nabla^2 \tilde{u} \begin{bmatrix} \tilde{u}(x, y, t) = \frac{1}{2} \left[\mathbf{u}(y) e^{i\alpha(x-ct)} + \mathbf{u}^*(y) e^{-i\alpha(x-c^*t)} \right] \\ \tilde{v}(x, y, t) = \frac{1}{2} \left[\mathbf{v}(y) e^{i\alpha(x-ct)} + \mathbf{v}^*(y) e^{-i\alpha(x-c^*t)} \right] \\ \frac{\partial \tilde{v}}{\partial t} + U \frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} = \mathbf{R} e^{-1} \nabla^2 \tilde{v} \end{bmatrix} \begin{bmatrix} \tilde{v}(x, y, t) = \frac{1}{2} \left[\mathbf{v}(y) e^{i\alpha(x-ct)} + \mathbf{v}^*(y) e^{-i\alpha(x-c^*t)} \right] \\ \tilde{v}(x, y, t) = \frac{1}{2} \left[\mathbf{p}(y) e^{i\alpha(x-ct)} + \mathbf{p}^*(y) e^{-i\alpha(x-c^*t)} \right] \end{bmatrix}
$$

or two differential equations capture, the normal modes of the normal modes of the normal modes, the real disturbance of the normal modes, the normal modes will be solved in the normal modes, the real disturbance of the no same dynamics. We will not derive them in full (try this as an exercise). can equality pressure the above. **Orr-Sommerfeld equation (1907-1908)**

$$
(U - c)\left(\frac{d^2\phi(y)}{dy^2} - \alpha^2\phi(y)\right) - \frac{d^2U}{dy^2}\phi(y) = \frac{1}{i\alpha \text{Re}}\left(\frac{d^4\phi(y)}{dy^4} - 2\alpha^2\frac{d^2\phi(y)}{dy^2} + \alpha^4\phi(y)\right)
$$

d2

⇣d²

INSTITUT PPRIME d*y*₂ (*y*)² (*y*)² (*y*)² (*y*)² (*y*)² (*y*)² (*y*)²

facilitated by the linearity of the system, the system, the system, the above PDEs can be reduced to ODEs can
The above PDEs can be reduced to ODEs can

9. Substitute into linearised equations of motion s of motion

$$
\begin{array}{ccc}\n\frac{\partial \tilde{u}}{\partial t} + U \frac{\partial \tilde{u}}{\partial x} + \frac{\partial V}{\partial y} & = 0 \\
\frac{\partial \tilde{u}}{\partial t} + U \frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y} \tilde{v} + \frac{\partial \tilde{p}}{\partial x} & = \mathrm{Re}^{-1} \nabla^2 \tilde{u} & \begin{array}{c}\n\tilde{u}(x, y, t) = \frac{1}{2} \big[\mathbf{u}(y) e^{i\alpha(x - ct)} + \mathbf{u}^*(y) e^{-i\alpha(x - c^*)} \big] \\
\tilde{v}(x, y, t) = \frac{1}{2} \big[\mathbf{v}(y) e^{i\alpha(x - ct)} + \mathbf{v}^*(y) e^{-i\alpha(x - c^*)} \big] \\
\frac{\partial \tilde{v}}{\partial t} + U \frac{\partial \tilde{v}}{\partial x} & + \frac{\partial \tilde{p}}{\partial y} & = \mathrm{Re}^{-1} \nabla^2 \tilde{v} & \begin{array}{c}\n\tilde{p}(x, y, t) = \frac{1}{2} \big[\mathbf{p}(y) e^{i\alpha(x - ct)} + \mathbf{p}^*(y) e^{-i\alpha(x - c^*)} \big] \\
\tilde{p}(x, y, t) = \frac{1}{2} \big[\mathbf{p}(y) e^{i\alpha(x - ct)} + \mathbf{p}^*(y) e^{-i\alpha(x - c^*)} \big] & \n\end{array}\n\end{array}
$$

where one or two differential equations capture, the normal modes, the normal modes, the real disturbance quantities \mathbf{R} same dynamics. We will not derive them in full (try this as an exercise). We will derive them in full (try thi
This as an exercise). We will derive them in full (try this as an exercise). We will derive the secrets of the $\frac{1}{2}$ can the above. The above the above the above the above. whose inviscible counterpart (1880) and Rayleigh equation (1880)

$$
(U-c)\left(\frac{d^2\mathbf{v}(y)}{dy^2} - \alpha^2\mathbf{v}(y)\right) - \frac{d^2U}{dy^2}\mathbf{v}(y) = 0
$$

Just as we show that the linearised earlier that the linearised earlier that the linearised in the linearised i
The linearised in the linearised equations can be formulated in the linearised in the linear state of the line

facilitated by the linearity of the system, the system, the system, the above PDEs can be reduced to ODEs can
The above PDEs can be reduced to ODEs can

⇣d²

(*y*)

9. Substitute into linearised equations of motion s of motion whose into investigate counterpart of the Rayleigh counterpart, is the Rayleigh contribution of the Rayleigh counterpart of the Rayleigh contribution of the Rayleigh counterpart of the Rayleigh contribution of the Rayleigh

$$
\frac{\partial \tilde{u}}{\partial t} + U \frac{\partial \tilde{u}}{\partial x} + \frac{\partial U}{\partial y} \tilde{v} + \frac{\partial \tilde{p}}{\partial x} = \text{Re}^{-1} \nabla^2 \tilde{u} \begin{bmatrix} \tilde{u}(x, y, t) = \frac{1}{2} \Big[\mathbf{u}(y) e^{i\alpha(x - ct)} + \mathbf{u}^*(y) e^{-i\alpha(x - c^*t)} \Big] \\ \tilde{v}(x, y, t) = \frac{1}{2} \Big[\mathbf{v}(y) e^{i\alpha(x - ct)} + \mathbf{v}^*(y) e^{-i\alpha(x - c^*t)} \Big] \\ \tilde{v}(x, y, t) = \frac{1}{2} \Big[\mathbf{v}(y) e^{i\alpha(x - ct)} + \mathbf{v}^*(y) e^{-i\alpha(x - c^*t)} \Big] \\ \tilde{v}(x, y, t) = \frac{1}{2} \Big[\mathbf{p}(y) e^{i\alpha(x - ct)} + \mathbf{p}^*(y) e^{-i\alpha(x - c^*t)} \Big] \end{bmatrix}
$$

where one or two differential equations capture, the normal modes, the normal modes, the real disturbance quantities \mathbf{R} same dynamics. We will not derive them in full (try this as an exercise). We will derive them in full (try thi
This as an exercise). We will derive them in full (try this as an exercise). We will derive the secrets of the $\frac{1}{2}$ can the above. The above the above the above the above. **Rayleigh equation (1880)** d2 wation (18)
Majah equation (18)

U

$$
(U-c)\left(\frac{d^2\phi(y)}{dy^2} - \alpha^2\phi(y)\right) - \frac{d^2U}{dy^2}\phi(y) = 0
$$

For bounded flows the boundary conditions require that (*y*) and ^d(*y*)

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d*y*

facilitated by the linearity of the system, the system, the system, the above PDEs can be reduced to ODEs can
The above PDEs can be reduced to ODEs can

10. Use boundary conditions, specify wavenumber or frequency and solve EIGENVALUE PROBLEM whose inviscid counterpart, derived over 25 years previous, is the Rayleight over 25 years previous, is the Ra

$$
(U-c)\left(\frac{d^2\mathbf{v}(y)}{dy^2} - \alpha^2\mathbf{v}(y)\right) - \frac{d^2U}{dy^2}\mathbf{v}(y) = 0
$$

Boundary conditions: v

- **- Bounded flow:** $-$ **Bounded fl** $\mathbf{v}(\mathbf{y}) = 0$ $\mathbf{v}(y)=0$
	- d
Dv - Unbounded flow: solution must be bounded
 \blacksquare

10. Use boundary conditions, specify wavenumber or frequency and solve EIGENVALUE PROBLEM e*ⁱ*(↵*r*+*i*↵*i*)*^x*e*i*(↵*rcr*+*ici*↵*r*+*icr*↵*i*+*i*2*ci*↵*i*)*^t* (93)

$$
(U-c)\left(\frac{d^2\mathbf{v}(y)}{dy^2} - \alpha^2\mathbf{v}(y)\right) - \frac{d^2U}{dy^2}\mathbf{v}(y) = 0
$$

The problem has been reduced to the above second-order ODE e↵*ix*+!*i^t* (98)

 $\overline{II}(a)$ in die verently the velocity streamfunction is used: $U(y)$ is the known base flow

> $\mathbf{V}(\mathcal{Y})$ is the unkn $\mathbf{v}(y)$ is the unknown radial structure of the perturbation
 C and α are unknown complex numbers

> $V(y)$ is the unknown radial structure of th
 C and α are unknown complex numbers !*ⁱ >* 0 (100)

10. Use boundary conditions, specify wavenumber or frequency and solve EIGENVALUE PROBLEM

$$
(U-c)\left(\frac{d^2\mathbf{v}(y)}{dy^2} - \alpha^2\mathbf{v}(y)\right) - \frac{d^2U}{dy^2}\mathbf{v}(y) = 0
$$

To solve the system: Written in the velocity streamfunction in the velocity streamfunction is used:

- **- Specify REAL WAVENUMBER** ⇣d² (*y*) ⌘ *U*
	- FREQUENCY is then a complex eigenvalue with eigenvector $\mathbf{v}(y)$ $\boldsymbol{\mu}$ 2 $\boldsymbol{\mu}$) $\boldsymbol{\mu}$ is then a complex eigenvalue with eigenvector $\mathbf{v}(y)$
		- **- TEMPORAL stability problem**

^d*y*² ↵²

^d*y*² (*y*)=0 (106)

10. Use boundary conditions, specify wavenumber or frequency and solve EIGENVALUE PROBLEM

$$
(U-c)\left(\frac{d^2\mathbf{v}(y)}{dy^2} - \alpha^2\mathbf{v}(y)\right) - \frac{d^2U}{dy^2}\mathbf{v}(y) = 0
$$

To solve the system: Written in the velocity streamfunction in the velocity streamfunction is used:

- **- Specify REAL FREQUENCY** ⇣d² (*y*) ⌘ *U*
	- **WAVENUMBER** is then a complex eigenvalue with eigenvector $\mathbf{v}(y)$ $\overline{\bf BER}$ is then a complex eigenvalue with eigenvector ${\bf v}(y)$
		- **- SPATIAL stability problem**

^d*y*² ↵²

^d*y*² (*y*)=0 (106)

3. The Squire transformation

Squire (1933) identified and exploited a similarity between the 2- and 3-D Orr-Sommerfeld equations, *i*↵Re ^d*y*⁴ 2(↵² ⁺ ² ^d*y*² + (↵² ⁺ ²

Consider a 3-D disturbance, to a base flow, $\,U(y)$, with polar wavenumber, $\mathbf{U} \cdot \mathbf{U} = \mathbf{U} \cdot \mathbf{U} = \mathbf{U} \cdot \mathbf{U} \cdot \mathbf{U} \cdot \mathbf{U} = \mathbf{U} \cdot \mathbf{U} \$ $\frac{d}{dx}$

$$
\tilde{\alpha} = \sqrt{\alpha_{3D} + \beta_{3D}}
$$

and which leads to an unstable solution of the 3-D Orr-Sommerfeld equation Consider a 3D disturbance, with polar wavenumber 2009 and ^a ^p⊥30 to 3 *D* and ^a 3 *D* and ^a 3 *D* and ^a 3 *D* to 3 D and a 3 D and a 4 D and a which leads to an unstable solution of the 3-D Off-Solutierield equ

$$
(U-c)\left(\frac{d^2\mathbf{v}(y)}{dy^2} - \tilde{\alpha}^2\mathbf{v}(y)\right) - \frac{d^2U}{dy^2}\mathbf{v}(y) = \frac{1}{i\alpha_{3D}\text{Re}_{3D}}\left(\frac{d^4\mathbf{v}(y)}{dy^4} - 2\tilde{\alpha}^2\frac{d^2\mathbf{v}(y)}{dy^2} + \tilde{\alpha}^4\mathbf{v}(y)\right)
$$

$$
(U - c)\left(\frac{d^2\mathbf{v}(y)}{dy^2} - \tilde{\alpha}^2\mathbf{v}(y)\right) - \frac{d^2U}{dy^2}\mathbf{v}(y) = \frac{1}{i\alpha_{3D}\text{Re}_{3D}}\left(\frac{d^4\mathbf{v}(y)}{dy^4} - 2\tilde{\alpha}^2\frac{d^2\mathbf{v}(y)}{dy^2} + \tilde{\alpha}^4\mathbf{v}(y)\right)
$$

Compare with 2-D Or-Sommerfeld equation

$$
(U - c)\left(\frac{d^2\mathbf{v}(y)}{dy^2} - \alpha_{2D}^2\mathbf{v}(y)\right) - \frac{d^2U}{dy^2}\mathbf{v}(y) = \frac{1}{i\alpha_{2D}\text{Re}_{2D}}\left(\frac{d^4\mathbf{v}(y)}{dy^4} - 2\alpha_{2D}^2\frac{d^2\mathbf{v}(y)}{dy^2} + \alpha_{2D}^4\mathbf{v}(y)\right)
$$

These equations have identical solutions if:

$$
\alpha_{2D} = \tilde{\alpha} = \sqrt{\alpha_{3D} + \beta_{3D}}
$$
\n
$$
\alpha_{2D} \text{Re}_{2D} = \alpha_{3D}
$$
\n
$$
\text{Re}_{2D} = \frac{\alpha_{3D}}{\cdot}
$$

Re2*^D* =

↵3*^D*

Re3*^D* =

and

$$
\alpha_{3D} + \beta_{3D}
$$
\n
$$
\alpha_{2D} \text{Re}_{2D} = \alpha_{3D} \text{Re}_{3D}
$$
\n
$$
\text{Re}_{2D} = \frac{\alpha_{3D}}{\alpha_{2D}} \text{Re}_{3D} = \frac{\alpha_{3D}}{\tilde{\alpha}} \text{Re}_{3D}
$$

(117)

I N S T I T U T P P R I M E ↵˜ Re3*^D* (120) Which means that for a growing 3D disturbance at Re3*^D* with ↵3*D*, 3*^D* (↵˜ =

3. The Squire transformation These two equations have identical solutions in the solution of the solutions of the s and

(*U c*)

These two equations have identical solutions if

⇣d²

$$
\alpha_{2D} = \tilde{\alpha} = \sqrt{\alpha_{3D} + \beta_{3D}}
$$
\n
$$
\alpha_{2D}\kappa_{2D} = \frac{\alpha_{3D}}{\alpha_{3D}}
$$

^d*y*² ↵2*^D*

v(*y*)

 \perp

$$
\alpha_{2D}\text{Re}_{2D} = \alpha_{3D}\text{Re}_{3D}
$$

$$
\text{Re}_{2D} = \frac{\alpha_{3D}}{\alpha_{2D}}\text{Re}_{3D} = \frac{\alpha_{3D}}{\tilde{\alpha}}\text{Re}_{3D}
$$

 $\frac{1}{2}$ + 2 (121) $\frac{1}{2}$ (121) $\frac{1}{2}$

Re3*^D* =

↵2*^D*

The call a standard and exploited a similar the 2- and 2- and 3-dimensional the 2- and 3-dimensional model and 3-
The 2- and 3-dimensional contract of the 2- and 3-dimensional model and 3-dimensional model and 3-dimensiona **urbance** ↵3*^D* ↵2*^D* and

and

 $\tilde{\alpha} = \sqrt{\alpha - 1}$ β defining the polar wavenumber $\tilde{\alpha}$ Re2*^D* =

प
प्रदेश संस्कृति संस्कृ
प्रदेश संस्कृति संस **P** $\overline{ }$ The **For any unstable 3D disturbance There exists an unstable 2D disturbance** $\boldsymbol{\mu}$ e exists an unstable zD disturbance

$$
\tilde{\alpha} = \sqrt{\alpha_{3D} + \beta_{3D}}
$$
\n
$$
\alpha_{2D} = \tilde{\alpha} \quad \text{at} \quad \text{Re}_{2D} = \frac{\alpha_{3D}}{\tilde{\alpha}} \text{Re}_{3D}
$$

p

EXECUTED INSTITUT PPRIME additional contract of the second s Which means that for a growing 3D disturbance at Re3*^D* with ↵3*D*, 3*^D* (↵˜ = Which means that for a growing 3D disturbance at Re3*^D* with ↵3*D*, 3*^D* (↵˜ = Which means that for a growing 3D disturbance at Re3*^D* with ↵3*D*, 3*^D* (↵˜ =

↵3*^D*

↵˜ Re3*^D* (120)

Squire's theorem: If an exact two-dimensional parallel flow admits an unstable 3-D disturbance for a certain value of the Reynolds number, it also admits an unstable 2-D disturbance at a lower Reynolds number

OR

Squire's theorem: To each unstable 3-D disturbance there corresponds a more unstable 2-D disturbance

OR

Squire's theorem: To obtain the minimum critical Reynolds number it is sufficient to consider only two-dimensional disturbances