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DÉPARTEMENT D2 – FLUIDES
THERMIQUE ET COMBUSTION

An introduction to hydrodynamic stability

Lecture 2: The governing equations for fluid instability

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Overview of lecture 1

1. A quick recap. of lecture 1

2. Rayleigh and Orr-Sommerfeld equations

3. The Squire transformation

1. Recap. of lecture 1

Determining stability via consideration of ENERGY

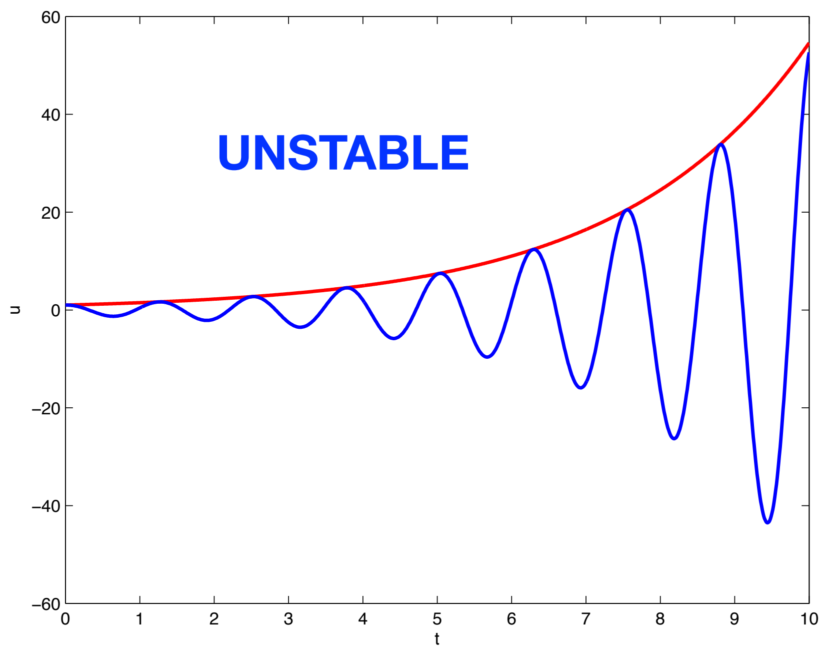
Determining stability via consideration of LINEAR DYNAMICS

ODE

$$\frac{du}{dt} = \lambda u$$

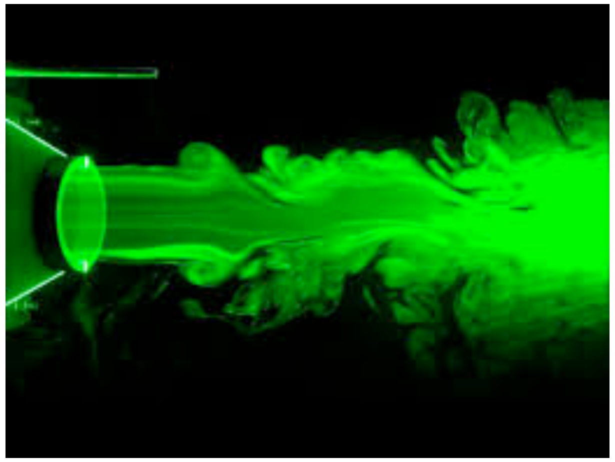
Solution

$$u(t) = Ae^{\lambda t}$$

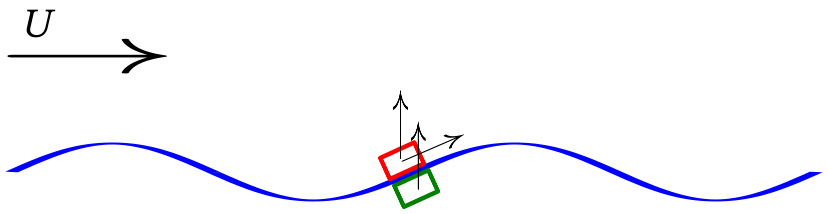


Connection with fluid mechanics?

Partial Differential Equations → **Solutions comprise a greater wealth of phenomena**



Kelvin-Helmholtz instability



Potential flow assumed above and below the vortex sheet: Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Disturbance *Ansatz*: velocity potential with normal modes:

$$\phi(x, z, t) = Ux + f(z)\mathrm{e}^{st+i\kappa x}$$

Leads to ODE for transverse structure: $$-\kappa^2 f(z) + \frac{d^2 f(z)}{dz^2} = 0$$

General solution:

$$f(z) = A_1 \mathrm{e}^{-\kappa z} + A_2 \mathrm{e}^{\kappa z}$$
$$f(z) = B_1 \mathrm{e}^{-\kappa z} + B_2 \mathrm{e}^{\kappa z}$$

Boundary and interface matching conditions: $$\eta(x,t) = \eta_0 \mathrm{e}^{\frac{1}{2}\kappa U t + i\kappa(x - \frac{1}{2}Ut)}$$

2. Rayleigh & Orr-Sommerfeld equations

The general approach for stabilty analysis can be understood by considering a simplified flow configuration:

- Parallel, 2D, shear flow,

which has surprisingly widespread applicability

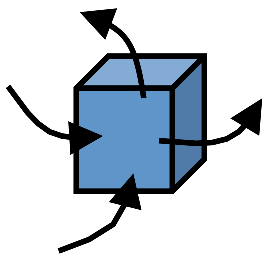
General approach:

1. Equations of motion (mass and momentum conservation)
2. Non-dimensionalisation
3. Identification of BASE-FLOW (steady laminar solution)
4. Decomposition of dependent variables into STEADY & FLUCTUATING quantities
5. Substitution into equations of motion
6. LINEARISATION (subtract base-flow equations; remove non-linear terms)
7. Reduce linearised equations to some compact form (often a single equation)
8. Express dependent variables in terms of NORMAL MODES
9. Introduction into linearised equation:
   - PDE system becomes a single ODE, but with too many unknowns
10. Specify a value for one of the unknowns (wavenumber for instance), solve for others: generally an EIGENVALUE PROBLEM

1. Equations of motion, in 2D, for incompressible, isentropic, flow:

Mass conservation:

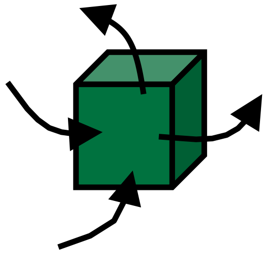
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$



Momentum conservation:

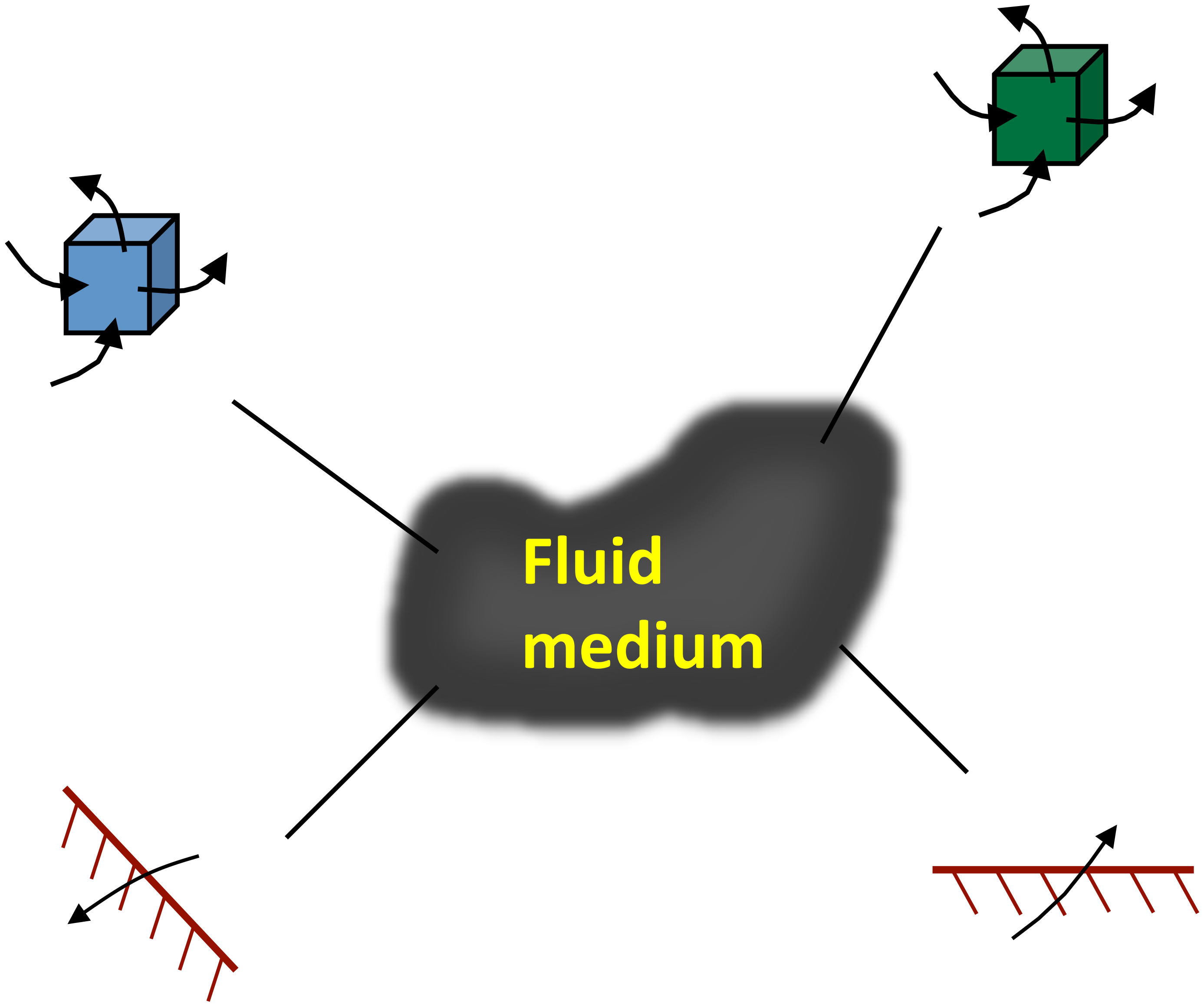
$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial p}{\partial x} = \mu\nabla^2 u$$

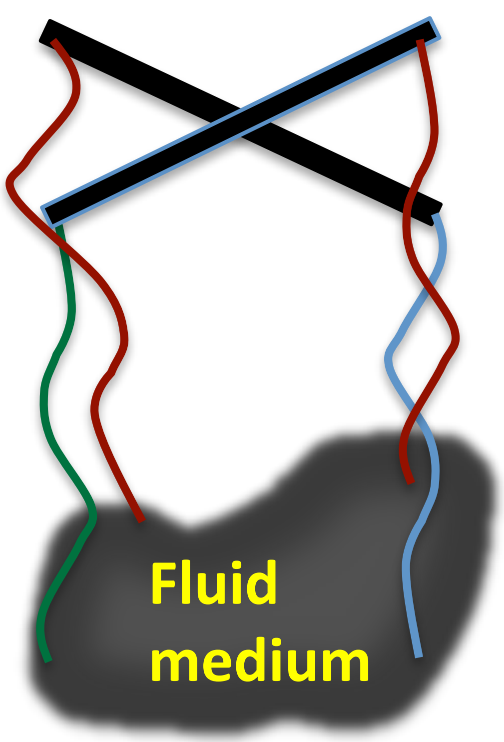
$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + \frac{\partial p}{\partial y} = \mu\nabla^2 v$$



Fluid motions constrained by:

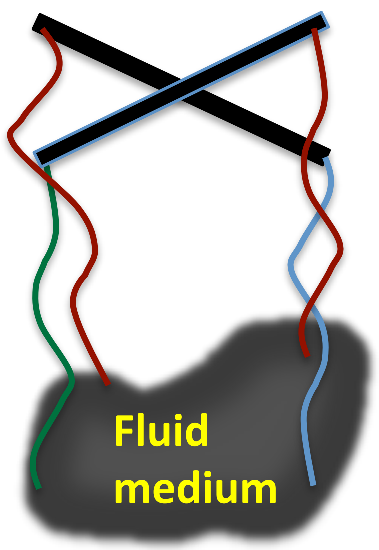
- Mass conservation
- Momentum conservation
- Boundary conditions





1. Equations of motion, in 2D, for incompressible, isentropic, flow:

Mass conservation:



$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Momentum conservation:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial p}{\partial x} = \mu\nabla^2 u$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + \frac{\partial p}{\partial y} = \mu\nabla^2 v$$

2. Non-dimensionalisation

$$u = \frac{u^+}{U_c} \qquad x = \frac{x^+}{L}$$

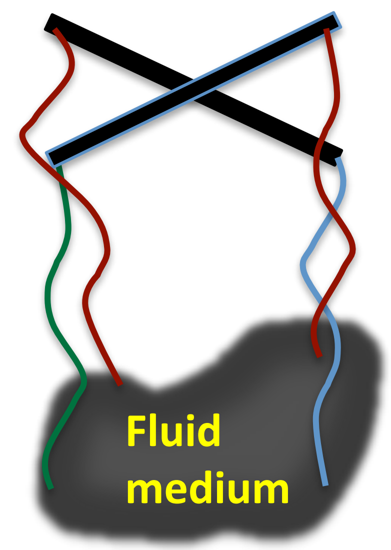
$$v = \frac{v^+}{U_c} \qquad y = \frac{y^+}{L}$$

$$p = \frac{p^+}{\rho U_c^2} \qquad t = \frac{t^+ U_c}{L}$$

1. Equations of motion, in 2D, for incompressible, isentropic, flow:
2. Non-dimensionalisation

Mass conservation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$



Momentum conservation:

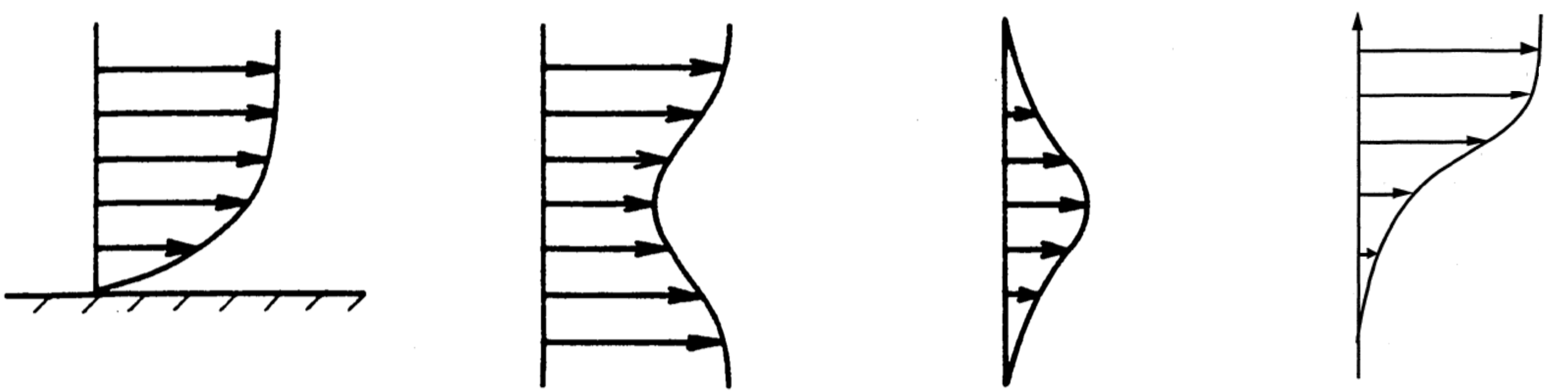
$$\mathrm{Re} = \frac{\rho L U_c}{\mu}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \mathrm{Re}^{-1}\nabla^2 u$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \mathrm{Re}^{-1}\nabla^2 v$$

3. Identification of BASE-FLOW

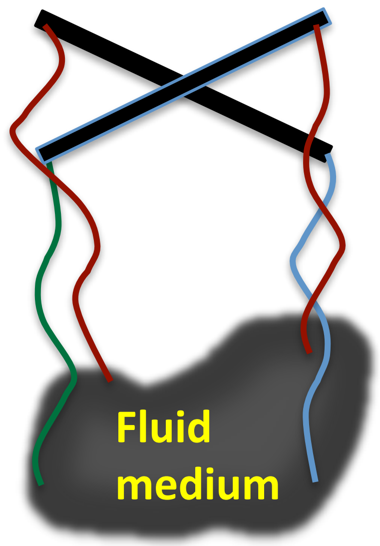
- Parallel and 2D (if flow changes slowly in some direction a locally parallel approximation is often adequate)

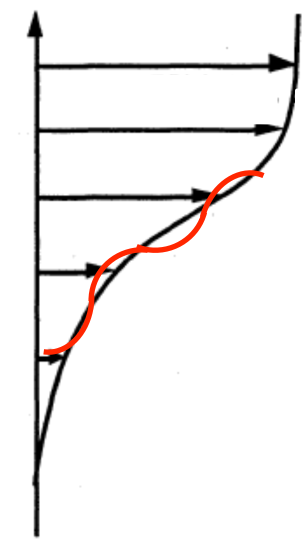
$$U(y)$$



Boundary layer   Wake   Jet   Shear-layer

4. Decomposition into STEADY & FLUCTUATING quantities





$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \mathrm{Re}^{-1}\nabla^2 u$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \mathrm{Re}^{-1}\nabla^2 v$$

$$u(x,y,t) = U(y) + \tilde{u}(x,y,t)$$
$$v(x,y,t) = \tilde{v}(x,y,t)$$
$$p(x,y,t) = P(x) + \tilde{p}(x,y,t)$$

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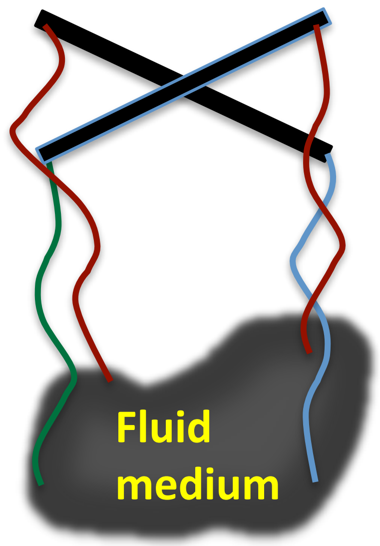
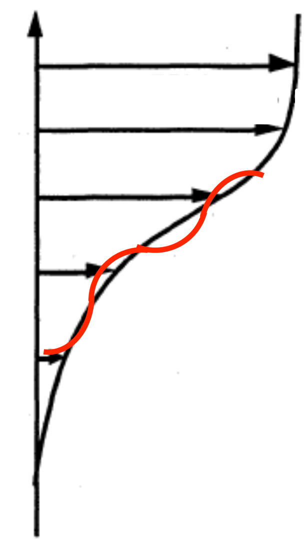
5. Substitute into equations of motion & subtract base-flow equations



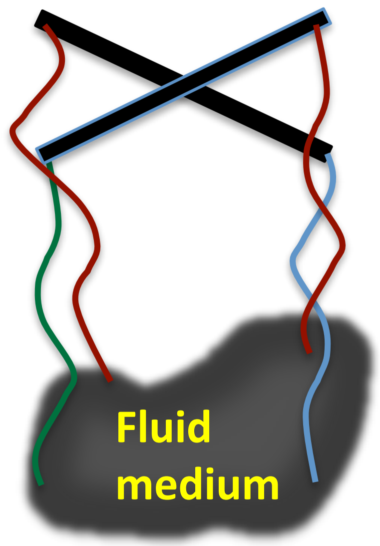


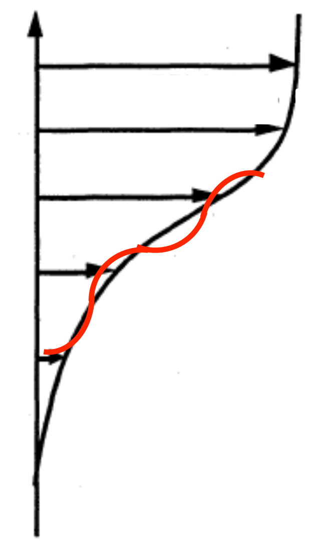
$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0$$

$$\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y}\tilde{v} + \frac{\partial \tilde{p}}{\partial x} + \left(\tilde{u}\frac{\partial \tilde{u}}{\partial x} + \tilde{v}\frac{\partial \tilde{u}}{\partial y}\right) = \mathrm{Re}^{-1}\nabla^2\tilde{u}$$

$$\frac{\partial \tilde{v}}{\partial t} + U\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} + \left(\tilde{u}\frac{\partial \tilde{v}}{\partial x} + \tilde{v}\frac{\partial \tilde{v}}{\partial y}\right) = \mathrm{Re}^{-1}\nabla^2\tilde{v}$$

6. LINEARISATION



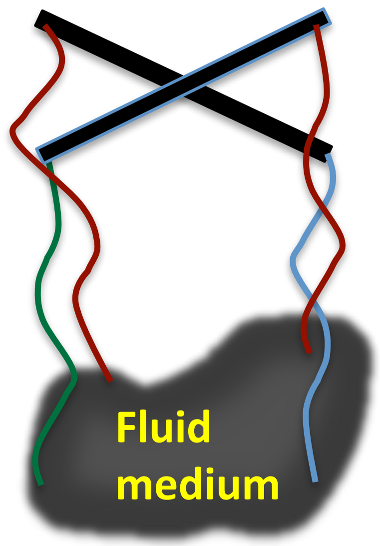


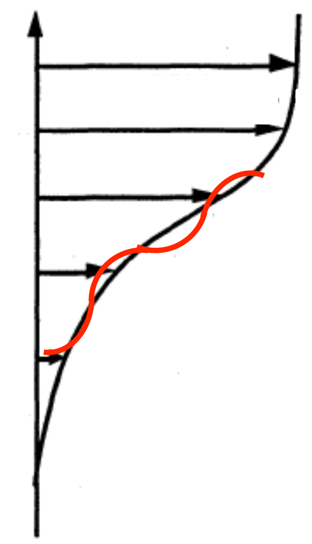
$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0$$

$$\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y}\tilde{v} + \frac{\partial \tilde{p}}{\partial x} + \cancelto{0}{\left(\tilde{u}\frac{\partial \tilde{u}}{\partial x} + \tilde{v}\frac{\partial \tilde{u}}{\partial y}\right)} = \mathrm{Re}^{-1}\nabla^2\tilde{u}$$

$$\frac{\partial \tilde{v}}{\partial t} + U\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} + \cancelto{0}{\left(\tilde{u}\frac{\partial \tilde{v}}{\partial x} + \tilde{v}\frac{\partial \tilde{v}}{\partial y}\right)} = \mathrm{Re}^{-1}\nabla^2\tilde{v}$$

6. LINEARISATION

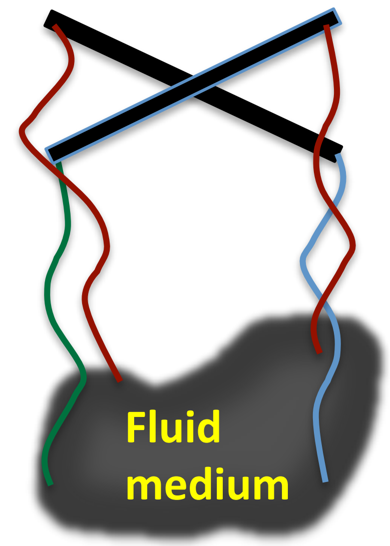


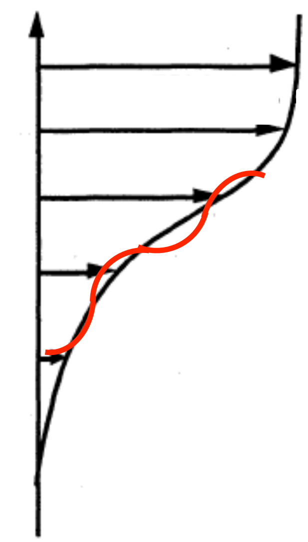


$$
\begin{aligned}
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= 0 \\
\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y}\tilde{v} + \frac{\partial \tilde{p}}{\partial x} &= \mathrm{Re}^{-1}\nabla^2\tilde{u} \\
\frac{\partial \tilde{v}}{\partial t} + U\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} &= \mathrm{Re}^{-1}\nabla^2\tilde{v}
\end{aligned}
$$

7. Reduction of linearised system to more COMPACT FORM

$$
\begin{aligned}
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= 0 \\
\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y}\tilde{v} + \frac{\partial \tilde{p}}{\partial x} &= \mathrm{Re}^{-1}\nabla^2\tilde{u} \\
\frac{\partial \tilde{v}}{\partial t} + U\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} &= \mathrm{Re}^{-1}\nabla^2\tilde{v}
\end{aligned}
$$





VELOCITY disturbance equation

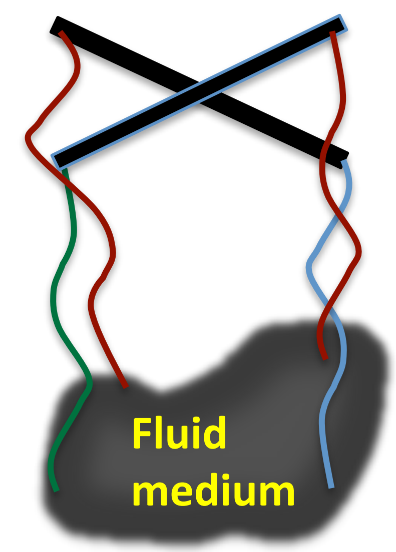
*

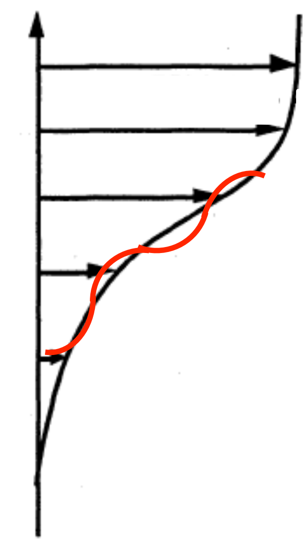
- Divergence of mom. eqs.
- Eliminate divergence-free terms
- Take Laplacian of v-mom. eq.
- Eliminate pressure

$$
\boxed{\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\tilde{v} - \frac{\mathrm{d}^2U}{\mathrm{d}y^2}\frac{\partial \tilde{v}}{\partial x} = \mathrm{Re}^{-1}\nabla^4\tilde{v}}
$$

7. Reduction of linearised system to more COMPACT FORM

$$
\begin{aligned}
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= 0 \\
\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y}\tilde{v} + \frac{\partial \tilde{p}}{\partial x} &= \mathrm{Re}^{-1}\nabla^2 \tilde{u} \\
\frac{\partial \tilde{v}}{\partial t} + U\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} &= \mathrm{Re}^{-1}\nabla^2 \tilde{v}
\end{aligned}
$$





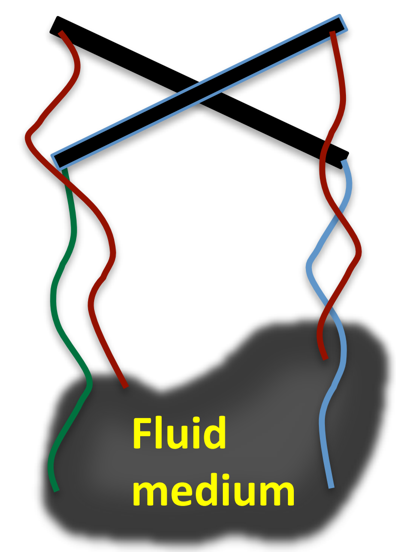
STREAMFUNCTION disturbance equation

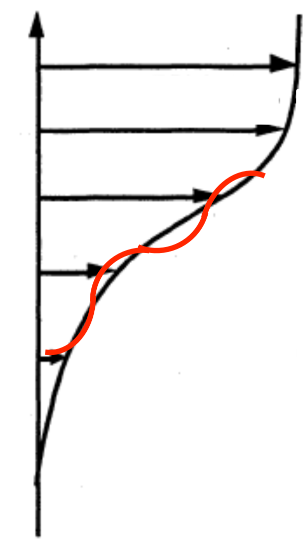
- Curl of momentum equations
- Substitute streamfunction (which automatically satisfies continuity equation)

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\Psi - \frac{\mathrm{d}^2 U}{\mathrm{d}y^2}\frac{\partial \Psi}{\partial x} = \mathrm{Re}^{-1}\nabla^4\Psi$$

7. Reduction of linearised system to more COMPACT FORM

$$
\begin{aligned}
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= 0 \\
\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y}\tilde{v} + \frac{\partial \tilde{p}}{\partial x} &= \mathrm{Re}^{-1}\nabla^2 \tilde{u} \\
\frac{\partial \tilde{v}}{\partial t} + U\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} &= \mathrm{Re}^{-1}\nabla^2 \tilde{v}
\end{aligned}
$$





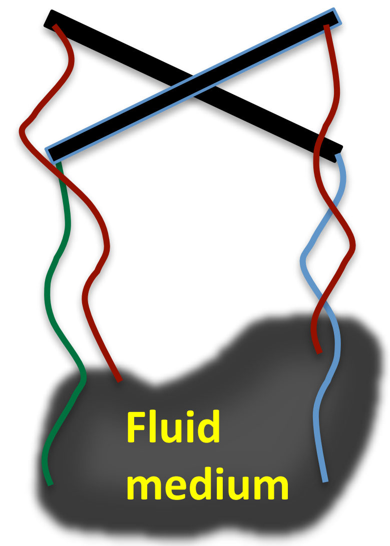
PRESSURE disturbance equation

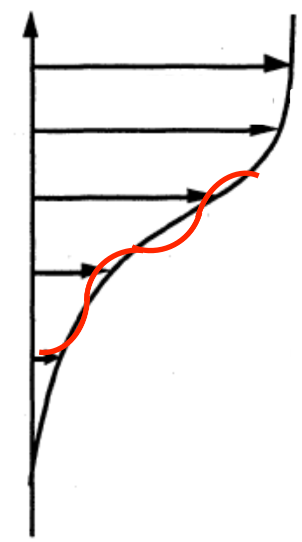
- Differentiate u- and v- mom. equations w.r.t. x & y
- Add them & simplify using continuity
- Material derivative of result
- Eliminate dv/dt using mom. eq.

$$
\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 \tilde{p} - 2\frac{\mathrm{d}U}{\mathrm{d}y}\frac{\partial \tilde{p}}{\partial x \partial y} = -2\frac{\mathrm{d}U}{\mathrm{d}y}\mathrm{Re}^{-1}\nabla^2\frac{\partial \tilde{v}}{\partial x}
$$

7. Reduction of linearised system to more COMPACT FORM

$$
\begin{aligned}
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= 0 \\
\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y}\tilde{v} + \frac{\partial \tilde{p}}{\partial x} &= \mathrm{Re}^{-1}\nabla^2 \tilde{u} \\
\frac{\partial \tilde{v}}{\partial t} + U\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} &= \mathrm{Re}^{-1}\nabla^2 \tilde{v}
\end{aligned}
$$





VORTICITY disturbance equation

- Curl of momentum equations
  which automatically eliminates pressure

$$
\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\tilde{\omega}_z - \frac{\mathrm{d}^2U}{\mathrm{d}y^2}\tilde{v} = \mathrm{Re}^{-1}\nabla^2\tilde{\omega}_z
$$

8. Express dependent variables as NORMAL MODES

- Decompose the *x* and *t* directions into Fourier modes

- Streamwise spatial structure expanded as spectrum of spatial modes (sines & cosines) characterised by their WAVENUMBERS

$$\alpha = \alpha_r + i\alpha_i$$

- Temporal structure (*t*-direction) expanded as spectrum of temporal modes (sines & cosines) characterised by their FREQUENCIES

$$\omega = \omega_r + i\omega_i$$

- Perturbations can be treated as a superposition of waves travelling at speed *c*:

$$\mathrm{e}^{i\alpha(x-ct)} \qquad c = c_r + ic_i$$

8. Express dependent variables as NORMAL MODES

Complex frequency is

$$\omega = \omega_r + i\omega_i$$

$$\omega_r = \alpha_r c_r - \alpha_i c_i$$

$$\omega_i = \alpha_r c_i + \alpha_i c_r$$

General disturbance

$$\mathrm{e}^{i\alpha(x-ct)} = \mathrm{e}^{i(\alpha_r+i\alpha_i)x}\mathrm{e}^{-i(\alpha_r+i\alpha_i)(c_r+ic_i)t}$$

Show that

$$= \mathrm{e}^{i(\alpha_r x-\omega_r t)}\mathrm{e}^{-\alpha_i x+\omega_i t}$$

8. Express dependent variables as NORMAL MODES

Complex frequency is

$$\omega = \omega_r + i\omega_i$$

$$\omega_r = \alpha_r c_r - \alpha_i c_i$$

$$\omega_i = \alpha_r c_i + \alpha_i c_r$$

General disturbance

$$\mathrm{e}^{i\alpha(x-ct)} = \mathrm{e}^{i(\alpha_r+i\alpha_i)x}\mathrm{e}^{-i(\alpha_r+i\alpha_i)(c_r+ic_i)t}$$

$$\mathrm{e}^{i(\alpha_r+i\alpha_i)x}\mathrm{e}^{-i(\alpha_r c_r+ic_i\alpha_r+ic_r\alpha_i+i^2c_i\alpha_i)t}$$

$$\mathrm{e}^{i(\alpha_r+i\alpha_i)x}\mathrm{e}^{-i(\alpha_r c_r+ic_i\alpha_r+ic_r\alpha_i-c_i\alpha_i)t}$$

Show that

$$\mathrm{e}^{i(\alpha_r+i\alpha_i)x}\mathrm{e}^{-i(\alpha_r c_r-c_i\alpha_i)t-i^2(c_i\alpha_r+c_r\alpha_i)t}$$

$$\mathrm{e}^{i(\alpha_r+i\alpha_i)x}\mathrm{e}^{-i\omega_r t+\omega_i t}$$

$$\mathrm{e}^{i\alpha_r x-\alpha_i x}\mathrm{e}^{-i\omega_r t+\omega_i t}$$

$$= \mathrm{e}^{i(\alpha_r x-\omega_r t)}\mathrm{e}^{-\alpha_i x+\omega_i t}$$

8. Express dependent variables as NORMAL MODES

General disturbance proportional to

$$\mathrm{e}^{i\alpha(x-ct)} = \underbrace{\mathrm{e}^{i(\alpha_r x-\omega_r t)}}_{\text{Space-time oscillation}}\underbrace{\mathrm{e}^{-\alpha_i x+\omega_i t}}_{\text{Space-time growth or decay}}$$

$\alpha_i < 0 \longrightarrow$ Exponential spatial growth

$\omega_i > 0 \longrightarrow$ Exponential temporal growth

8. Express dependent variables as NORMAL MODES

General disturbance proportional to

$$\mathrm{e}^{i\alpha(x-ct)} = \mathrm{e}^{i(\alpha_r x-\omega_r t)}\mathrm{e}^{-\alpha_i x+\omega_i t}$$

$\alpha_i < 0$ → Exponential spatial growth

$\omega_i > 0$ → Exponential temporal growth

Temporal stability $\alpha_i = 0$ and ω complex

Spatial stability $\omega_i = 0$ and α complex

9. Substitute into linearised equations of motion

$$\boxed{\begin{aligned} &\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &&= 0 \\ &\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y}\tilde{v} + \frac{\partial \tilde{p}}{\partial x} &&= \mathrm{Re}^{-1}\nabla^2\tilde{u} \\ &\frac{\partial \tilde{v}}{\partial t} + U\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} &&= \mathrm{Re}^{-1}\nabla^2\tilde{v} \end{aligned}}$$

$$\tilde{u}(x,y,t) = \frac{1}{2}\left[\mathbf{u}(y)\mathrm{e}^{i\alpha(x-ct)} + \mathbf{u}^*(y)\mathrm{e}^{-i\alpha(x-c^*t)}\right]$$
$$\tilde{v}(x,y,t) = \frac{1}{2}\left[\mathbf{v}(y)\mathrm{e}^{i\alpha(x-ct)} + \mathbf{v}^*(y)\mathrm{e}^{-i\alpha(x-c^*t)}\right]$$
$$\tilde{p}(x,y,t) = \frac{1}{2}\left[\mathbf{p}(y)\mathrm{e}^{i\alpha(x-ct)} + \mathbf{p}^*(y)\mathrm{e}^{-i\alpha(x-c^*t)}\right]$$

Orr-Sommerfeld equation (1907-1908)

$$\boxed{(U-c)\left(\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2} - \alpha^2\mathbf{v}(y)\right) - \frac{\mathrm{d}^2U}{\mathrm{d}y^2}\mathbf{v}(y) = \frac{1}{i\alpha\mathrm{Re}}\left(\frac{\mathrm{d}^4\mathbf{v}(y)}{\mathrm{d}y^4} - 2\alpha^2\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2} + \alpha^4\mathbf{v}(y)\right)}$$

9. Substitute into linearised equations of motion

$$
\begin{aligned}
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= 0 \\
\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y}\tilde{v} + \frac{\partial \tilde{p}}{\partial x} &= \mathrm{Re}^{-1}\nabla^2 \tilde{u} \\
\frac{\partial \tilde{v}}{\partial t} + U\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} &= \mathrm{Re}^{-1}\nabla^2 \tilde{v}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{u}(x,y,t) &= \frac{1}{2}\left[\mathbf{u}(y)\mathrm{e}^{i\alpha(x-ct)} + \mathbf{u}^*(y)\mathrm{e}^{-i\alpha(x-c^*t)}\right] \\
\tilde{v}(x,y,t) &= \frac{1}{2}\left[\mathbf{v}(y)\mathrm{e}^{i\alpha(x-ct)} + \mathbf{v}^*(y)\mathrm{e}^{-i\alpha(x-c^*t)}\right] \\
\tilde{p}(x,y,t) &= \frac{1}{2}\left[\mathbf{p}(y)\mathrm{e}^{i\alpha(x-ct)} + \mathbf{p}^*(y)\mathrm{e}^{-i\alpha(x-c^*t)}\right]
\end{aligned}
$$

Orr-Sommerfeld equation (1907-1908)

$$
(U-c)\left(\frac{\mathrm{d}^2\phi(y)}{\mathrm{d}y^2} - \alpha^2\phi(y)\right) - \frac{\mathrm{d}^2U}{\mathrm{d}y^2}\phi(y) = \frac{1}{i\alpha\mathrm{Re}}\left(\frac{\mathrm{d}^4\phi(y)}{\mathrm{d}y^4} - 2\alpha^2\frac{\mathrm{d}^2\phi(y)}{\mathrm{d}y^2} + \alpha^4\phi(y)\right)
$$

9. Substitute into linearised equations of motion

$$\boxed{\begin{aligned} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= 0 \\ \frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y}\tilde{v} + \frac{\partial \tilde{p}}{\partial x} &= \mathrm{Re}^{-1}\nabla^2 \tilde{u} \\ \frac{\partial \tilde{v}}{\partial t} + U\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} &= \mathrm{Re}^{-1}\nabla^2 \tilde{v} \end{aligned}}$$

$$\begin{aligned} \tilde{u}(x,y,t) &= \frac{1}{2}\left[\mathbf{u}(y)\mathrm{e}^{i\alpha(x-ct)} + \mathbf{u}^*(y)\mathrm{e}^{-i\alpha(x-c^*t)}\right] \\ \tilde{v}(x,y,t) &= \frac{1}{2}\left[\mathbf{v}(y)\mathrm{e}^{i\alpha(x-ct)} + \mathbf{v}^*(y)\mathrm{e}^{-i\alpha(x-c^*t)}\right] \\ \tilde{p}(x,y,t) &= \frac{1}{2}\left[\mathbf{p}(y)\mathrm{e}^{i\alpha(x-ct)} + \mathbf{p}^*(y)\mathrm{e}^{-i\alpha(x-c^*t)}\right] \end{aligned}$$

Rayleigh equation (1880)

$$\boxed{(U-c)\left(\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2} - \alpha^2\mathbf{v}(y)\right) - \frac{\mathrm{d}^2U}{\mathrm{d}y^2}\mathbf{v}(y) = 0}$$

9. Substitute into linearised equations of motion

$$\boxed{\begin{aligned}
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= 0\\
\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \frac{\mathrm{d}U}{\mathrm{d}y}\tilde{v} + \frac{\partial \tilde{p}}{\partial x} &= \mathrm{Re}^{-1}\nabla^2\tilde{u}\\
\frac{\partial \tilde{v}}{\partial t} + U\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{p}}{\partial y} &= \mathrm{Re}^{-1}\nabla^2\tilde{v}
\end{aligned}}$$

$$\begin{aligned}
\tilde{u}(x,y,t) &= \frac{1}{2}\left[\mathbf{u}(y)\mathrm{e}^{i\alpha(x-ct)} + \mathbf{u}^*(y)\mathrm{e}^{-i\alpha(x-c^*t)}\right]\\
\tilde{v}(x,y,t) &= \frac{1}{2}\left[\mathbf{v}(y)\mathrm{e}^{i\alpha(x-ct)} + \mathbf{v}^*(y)\mathrm{e}^{-i\alpha(x-c^*t)}\right]\\
\tilde{p}(x,y,t) &= \frac{1}{2}\left[\mathbf{p}(y)\mathrm{e}^{i\alpha(x-ct)} + \mathbf{p}^*(y)\mathrm{e}^{-i\alpha(x-c^*t)}\right]
\end{aligned}$$

Rayleigh equation (1880)

$$\boxed{(U-c)\left(\frac{\mathrm{d}^2\phi(y)}{\mathrm{d}y^2} - \alpha^2\phi(y)\right) - \frac{\mathrm{d}^2U}{\mathrm{d}y^2}\phi(y) = 0}$$

10. Use boundary conditions, specify wavenumber or frequency and solve EIGENVALUE PROBLEM

$$\boxed{(U-c)\left(\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2}-\alpha^2\mathbf{v}(y)\right)-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\mathbf{v}(y)=0}$$

Boundary conditions:

- **Bounded flow:** $\mathbf{v}(y)=0$

- **Unbounded flow: solution must be bounded**

10. Use boundary conditions, specify wavenumber or frequency and solve
EIGENVALUE PROBLEM

$$\boxed{(U-c)\left(\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2}-\alpha^2\mathbf{v}(y)\right)-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\mathbf{v}(y)=0}$$

The problem has been reduced to the above second-order ODE

$U(y)$ is the known base flow

$\mathbf{v}(y)$ is the unknown radial structure of the perturbation

c and α are unknown complex numbers

Temporal stability	$\alpha_i = 0$ and ω complex
Spatial stability	$\omega_i = 0$ and α complex

10. Use boundary conditions, specify wavenumber or frequency and solve EIGENVALUE PROBLEM

$$\boxed{(U-c)\left(\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2}-\alpha^2\mathbf{v}(y)\right)-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\mathbf{v}(y)=0}$$

To solve the system:

- **Specify REAL WAVENUMBER**
  - **FREQUENCY is then a complex eigenvalue with eigenvector $\mathbf{v}(y)$**
    - **TEMPORAL stability problem**

10. Use boundary conditions, specify wavenumber or frequency and solve EIGENVALUE PROBLEM

$$\boxed{(U-c)\left(\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2}-\alpha^2\mathbf{v}(y)\right)-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\mathbf{v}(y)=0}$$

To solve the system:

- Specify REAL FREQUENCY
  - WAVENUMBER is then a complex eigenvalue with eigenvector $\mathbf{v}(y)$
    - SPATIAL stability problem

3. The Squire transformation

Squire (1933) identified and exploited a similarity between the 2- and 3-D Orr-Sommerfeld equations,

Consider a 3-D disturbance, to a base flow, $U(y)$, with polar wavenumber,

$$\tilde{\alpha} = \sqrt{\alpha_{3D} + \beta_{3D}}$$

and which leads to an unstable solution of the 3-D Orr-Sommerfeld equation

$$\boxed{(U - c)\left(\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2} - \tilde{\alpha}^2\mathbf{v}(y)\right) - \frac{\mathrm{d}^2U}{\mathrm{d}y^2}\mathbf{v}(y) = \frac{1}{i\alpha_{3D}\mathrm{Re}_{3D}}\left(\frac{\mathrm{d}^4\mathbf{v}(y)}{\mathrm{d}y^4} - 2\tilde{\alpha}^2\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2} + \tilde{\alpha}^4\mathbf{v}(y)\right)}$$

3. The Squire transformation

$$(U-c)\left(\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2}-\tilde{\alpha}^2\mathbf{v}(y)\right)-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\mathbf{v}(y)=\frac{1}{i\alpha_{3D}\mathrm{Re}_{3D}}\left(\frac{\mathrm{d}^4\mathbf{v}(y)}{\mathrm{d}y^4}-2\tilde{\alpha}^2\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2}+\tilde{\alpha}^4\mathbf{v}(y)\right)$$

Compare with 2-D Orr-Sommerfeld equation

$$(U-c)\left(\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2}-{\alpha_{2D}}^2\mathbf{v}(y)\right)-\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\mathbf{v}(y)=\frac{1}{i\alpha_{2D}\mathrm{Re}_{2D}}\left(\frac{\mathrm{d}^4\mathbf{v}(y)}{\mathrm{d}y^4}-2{\alpha_{2D}}^2\frac{\mathrm{d}^2\mathbf{v}(y)}{\mathrm{d}y^2}+{\alpha_{2D}}^4\mathbf{v}(y)\right)$$

These equations have identical solutions if:

$$\alpha_{2D}=\tilde{\alpha}=\sqrt{\alpha_{3D}+\beta_{3D}}$$

$$\alpha_{2D}\mathrm{Re}_{2D}=\alpha_{3D}\mathrm{Re}_{3D}$$

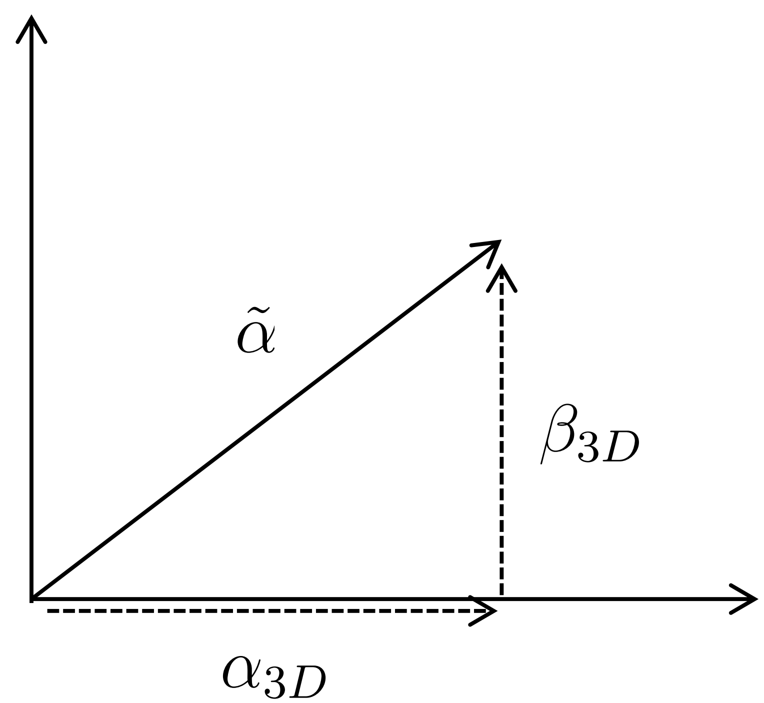
$$\mathrm{Re}_{2D}=\frac{\alpha_{3D}}{\alpha_{2D}}\mathrm{Re}_{3D}=\frac{\alpha_{3D}}{\tilde{\alpha}}\mathrm{Re}_{3D}$$

3. The Squire transformation

$$\alpha_{2D} = \tilde{\alpha} = \sqrt{\alpha_{3D} + \beta_{3D}}$$

$$\alpha_{2D}\mathrm{Re}_{2D} = \alpha_{3D}\mathrm{Re}_{3D}$$

$$\mathrm{Re}_{2D} = \frac{\alpha_{3D}}{\alpha_{2D}}\mathrm{Re}_{3D} = \frac{\alpha_{3D}}{\tilde{\alpha}}\mathrm{Re}_{3D}$$



For any unstable 3D disturbance

$$\tilde{\alpha} = \sqrt{\alpha_{3D} + \beta_{3D}}$$

There exists an unstable 2D disturbance

$\alpha_{2D} = \tilde{\alpha}$ **at** $\mathrm{Re}_{2D} = \dfrac{\alpha_{3D}}{\tilde{\alpha}}\mathrm{Re}_{3D}$

Squire's theorem: If an exact two-dimensional parallel flow admits an unstable 3-D disturbance for a certain value of the Reynolds number, it also admits an unstable 2-D disturbance at a lower Reynolds number

OR

Squire's theorem: To each unstable 3-D disturbance there corresponds a more unstable 2-D disturbance

OR

Squire's theorem: To obtain the minimum critical Reynolds number it is sufficient to consider only two-dimensional disturbances