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CNRS–UPR–3346 *•* UNIVERSITE DE POITIERS ´ *•* ENSMA

DÉPARTEMENT D2 – FLUIDES THERMIQUE ET COMBUSTION

December 11, 2012 **An introduction to hydrodynamic stability**

Cambridge University Department of Engineering **Trumping 1: General introduction**

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It is my pleasure to provide the provide this approximate the Dr. Agarwale of Dr. Agarwale of Dr. Agarwale to the Dr. Agarwale of Dr. Agarwal

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Who am I? Where do I come from?

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Lecture 1: General introduction

- Overview of different fluid instabilities
- Basic notions of stability
- Energy approach versus direct consideration of linear dynamics
- Shear-flow (Kelvin-Helmholtz instability)

Lecture 2: The governing equations for fluid instability

- Rayleigh equation
- Orr-Sommerfeld equation
- The Squire transformation

Lecture 3: Numerical methods

- Solving eigenvalue problems
- Inviscid temporal instability of a 2D mixing layer

Lecture 4: Viscous instability

- Viscosity and stability of plane Poiseuille flow
- Orr-Sommerfeld solution for mixing layer
- Boundary-layer instability
- Rayleigh's inflection-point theorem

Lecture 5: The spatial and spatiotemporal stability problems

- The linearised equations in full form
- The 2D mixing layer
- The compressible round jet
- Spatiotemporal stability
- Non-parallel flows

Lecture 6: Non-modal instability

- The enigma of pipe flow
- The Orr-Sommerfeld-Squire system
- The initial-value problem
- Non-normality and transient growth

Lecture 7: Beyond the critical point

- Rayleigh-Bénard convection
- State-space representation of dynamics systems
- Local bifurcation theory

Lecture 8: Weakly non-linear stability

- The Eckhaus equation
- The Stuart-Landau equation
- The Ginzburg-Landau equation

Lecture 9: Linear stability of fully turbulent flows

- Wavepackets and turbulent jet noise*

- Broad range of instability phenomena
- Importance applications

2. Why study hydrodynamic stability?

3. Basic notions of stability

- Stable and unstable systems
- Energy consideration
- Linear dynamics

4. Shear-flow (Kelvin-Helmholtz) instability

i. Kelvin-Helmholtz shear-flow instability

ii. Poisueille flow - Reynolds experiment

iii. Rayleigh-Plateau capillary instability

iv. Taylor-Couette centrifugal instability

v. Rayleigh-Bénard convective instability

vi. Rayleigh-Taylor interface instability (in stratified fluid)

vii. Tollmien-Schlichting viscous instability in wall-bounded flow

viii. Von-Karman wake instability

Rayleigh

Taylor

Helmholtz

Kelvin

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i. Kelvin-Helmholtz shear-flow instability

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viii. Von-Karman wake instability

Couette

Plateau

Sommerfeld

Von Karman

Kelvin-Helmholtz shear-flow instability

I N S T I T U T P P R I M E

Underpins transition in all inflectional shear-flows

Jets, wakes, shear-layers, jets in cross flow, separation from bluff bodies, separation from airfoil,…

Underpins coherent structures in fully turbulent inflectional shear flows

Important in aerodynamically generated noise (jet noise).

Von-Karman wake instability

Von-Karmen wake instability

I N S T I T U T P P R I M E

Underpins drag induced by bluff bodies, landing gear of aircraft, for example

Unsteady loads on offshore structures

Resonance in heat exchangers

Important in aerodynamically generated noise - landing-gear noise

Rayleigh-Plateau capillary instability

Plateau-Rayleigh capillary instability

I N S T I T U T P P R I M E

Flow from a tap

Ink-jet printers

- **Promote drop formation**
	- **Droplets charged electrically**
	- **Guided using electric field**
- **Essential that droplets have the same mass**

Spinning of nylon

- Important to prevent droplet formation to obtain constant-diameter fibres

Fluid-injection systems: atomisation important for combustion efficiency

Taylor-Couette instability

Lubrification in journal bearings

Rotating filtration: extracting plasma from blood, water purification,…

Magnetohydrodynamics: magnetic fields of planets

Rotational viscometers

Rayleigh-Bénard instability

Rayleigh-Bénard instability

Astrophysics - heat/energy transfer in outer atmosphere of stars

Geophysics - movement of Earth's mantle

Atmospheric science: weather systems, including long terms effects (ice-ages)

Solar energy systems

Nuclear systems

Material processing

Rayleigh-Taylor instability between two stable stratifications

Megan Davies Wykes and Stuart Dalziel DAMTP, University of Cambridge, UK

Heat transfer

Inertial confinement fusion

Nuclear bombs

Supernova explosions

Solar coronas

Lava lamp

Tollmien-Schlichting instability

I N S T I T U T P P R I M E

Transition of boundary layers

Drag reduction on wings - laminar-wing projects

2. Why study hydrodynamic stability?

In fluid mechanics, clear understanding is the exception rather than the rule

Turbulence is poorly understood due to nature of governing equations:

- **non-linear, in 4 dimensions, 6 dependent variables,…**
- **We can obtain approximate solutions using (very) large computers**
- **but solution does not imply understanding**

Hydrodynamic stability theory & analysis:

- **Direct and relatively complete understanding is available:**
	- **linearity, analytical solutions**
- **The magic of fluid mechanics is here most accessible:**
	- **the videos are visually striking**
	- **understanding makes them even better**
- **Surprisingly pertinent in many fully turbulent flows…**

I N S T I T U T P P R I M E

3. Some basic notions of stability

Unstable systems

Stable systems

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What have those systems got to do with fluid mechanics?

We can ask of a fluid system, be it still or in motion:

- What will be its response to an infinitesimally small perturbation?

We can consider the problem in terms of:

- **Energy states**
- **Linear dynamics**

Questions typically asked in a stability analysis:

- **1. Is system STABLE or UNSTABLE?**
- **2. How does this state change as some parameter is changed? - typically the Reynolds number in shear-flow problems**
- **3. How will system behave in response to a small perturbation? - What are its DYNAMICS?**

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I N S T I T U T P P R I M E

Energy consideration

Energy consideration

If perturbed system has MORE energy than steady state:

- **- Energy required to maintain perturbation**
- **- System is STABLE**

Stability corresponds to a state of minimal energy

Energy consideration

If perturbed system has LESS energy than steady state:

- **- No external energy required to amplify perturbation**
- **- System is UNSTABLE**

An unstable system has maximal energy and will release this in response to an infinitesimal perturbation so as to incline to a lower energy state.

Consideration of linear dynamics

Linearised equations of motion for the system, which could be:

- **- a pendulum,**
- **- a bouncing ball,**
- **- a rocking boat,**
- **- a flowing fluid,**
- **- …**

The system has general solutions of the system of the s
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dt

Sketch the dynamics that correspond to the following scenarios: a $\frac{1}{2}$ *d* $\frac{1}{2}$ *i* $\frac{1}{2}$ mamics that correspond to the following scenarios:

thoroughly before proceeding with the main material of the material of the main material of these lectures. Th
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= *u* (2)

Consider a spring-mass-damper system, governed by e.g. many vibrational systems **Consider a spring-mass-damper system, governed by EVA is a set of the system** system, governed by e.g. many vibrational systems

$$
\frac{d^2u}{dt^2} + 2\beta \frac{du}{dt} + \gamma u = 0
$$

Substitute e*^t* gives characteristics equation $\bm{\textsf{Perturbations}} \hspace{0.2cm} u(t) \hspace{0.2cm} \textsf{proportional to} \hspace{0.2cm} \bm{\textsf{e}}^{\lambda t}$ Substitute e*^t* gives characteristics equation $2 \div$ turbations $\omega(v)$ $\partial_t \lambda t$ ϵ

> Obtain characteristic equation: $\lambda^2 + 2\beta\lambda + \gamma = 0$ 1*,*² ⁼ *[±]* ^p² (13)

> > 1
1
1

If the real part of either *s*¹ or *s*² is positive

$$
\beta^2 > \gamma \qquad \lambda_{1,2} = -\beta \pm \sqrt{\beta^2 - \gamma}
$$

$$
\beta^2 < \gamma \qquad \lambda_{1,2} = -\beta \pm i\sqrt{\gamma - \beta^2}
$$

General solution

$$
u(t) = A e^{\lambda_1 t} + B e^{\lambda_2 t}
$$

solutions and the techniques used to obtain the techniques used to obtain the techniques indispensable indispen

 2×10^{-10} and 2×10^{-10}

Linear hydrodynamic stability theory is intimately tied to the properties of the **System is UNSTABLE if at least one root has positive real part** θ is the social continuous in θ

his is the case if either $\quad \beta < 0 \quad$ and/or $\quad \gamma < 0$ This is the case if either $\;\;\beta < 0\;\;$ and/or $\;\;\gamma < 0$ $s \sim$ and the techniques to obtain the techniques used to obtain the techniques individuals in \mathcal{L}

1*,*² ⁼ *[±]* ^p² (13)

2 A quick revision of differential equations of the contract o **What is the connection with fluid dynamics?** where *a* and *b* are real and positive.

thoroughly before proceeding with the main material of the material of the main material of these lectures. Th
The material of the material o

= *u* (2)

Summary Summary Either 1 or 2 is positive to either 1 Stability of a system can be considered

1.3 Summary

Stability of a system can be assessed by: Stability of *•* Either by considering the energy change *E* when system is perturbed

1. Considering the energy change when system is perturbed **•• Considering the energy change when system is perturb** *change when system is perturbed* **1. Considering the energy change when system is perturbed** to analysis of the source of \mathbf{r}

, bonsidering the linear dynamics, solutions \mathbf{C}^{new} , Considering the linear dynamics, solutions ϵ stable – If *E <* 0, energy is released from the system when it is perturbed, **2. Considering the linear dynamics, solutions** 2. Considering the linear dynamics, solutions $e^{\lambda t}$

4. Kelvin-Helmholtz shear-flow instability

Fluid-velocity gradient normal to flow direction

Encountered in a wide range of flows of engineering interest:

- **- Jets,**
- **- Wakes,**
- **- Boundary layers,**
- **- Mixing layers,**
- **- …**

Basic mechanism can be understood in a simplified configuration

- **- All vorticity concentrated on a line vortex sheet**
- **- Introduce a small stationary, wavy disturbance:**

$$
\overbrace{\qquad\qquad \qquad }^{A} \qquad \overbrace{\qquad\qquad }^{C} \qquad \qquad }
$$

Flow accelerates at the crests, A & D, where streamlines converge,

Flow decelerates at the troughs, B & C, where streamlines diverge,

Tangential flow speed:

- **greater at A than at B,**
- **greater at D than at C,**

Bernoulli says:

- **pressure at A less than at B**
- **force exerted from B to A and from C to D**
- **wave amplitude increases**
- **pressure difference increases**
- **normal force increases**
- **wave amplitude increases**

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Analysis using equations of motion is required for a complete description is required for a complete description of the flow of the flow

@*x*

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⁺ *^U* @

@*y*

@*z*

@*x*² *i*

)r² @²*^U*

@² @*x*² = 0 (14) **Above the vortex sheet we have potential flow; disturbance** *Ansatz***:**

²

⁼ ²

@*t*

$$
\phi(x, z, t) = Ux + f(z)e^{st + i\kappa x} \qquad z > \eta(x, t)
$$

²

v = 0 (11)

Potential flow -> Laplace's equation: 1.1 Potential flow of the state of the s

$$
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0
$$

@*x*

*d*²

 $+$

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@*z*

*dz*² = 0 (16)

x (poten *d*² *d* $\frac{d}{dx}$ *du (16)* $\frac{d}{dx}$ (*16)* $\frac{d}{dx}$ (*cx*) *si* **dz**
2008 Aposta Wilmotential atrove the startering ⇣ *^f*(*z*) + *^d*²*f*(*z*) **twice with respect to x, y & z** Differentiate disturbance Ansatz, \\\$(po\tent\)al above the v)ertex sh

@*t*

²

$$
-\kappa^2 f(z) + \frac{d^2 f(z)}{dz^2} = 0
$$

²

^f(*z*) + *^d*²*f*(*z*)

A second-order ODE constraining the transverse structure of the flow. he tra
 $\overline{\mathbf{a}}$ nsverse structur $\overline{\mathbf{b}}$ e of the flow.

and the flow set \sim 0 \sim 0 *i* @*x*

$$
-\kappa^2 f(z) + \frac{d^2 f(z)}{dz^2} = 0
$$

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^f(*z*) + *^d*²*f*(*z*)

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Second-order ODE has general solution:
 *f***(***z***)** $\frac{1}{2}$ $\frac{1}{2}$ *f*(*z*) \overline{a} *f*(*z*) \overline{b} \overline{c} \overline{d} \overline{d} \overline{c} \overline{c} \overline{c} \overline{c} \overline{c} $\overline{c$

²

$$
f(z) = A_1 e^{-\kappa z} + A_2 e^{\kappa z} \qquad z > \eta(x, t)
$$

$$
f(z) = B_1 e^{-\kappa z} + B_2 e^{\kappa z} \qquad z < \eta(x, t)
$$

^f(*z*) + *^d*²*f*(*z*)

⌘

e*st*+*i^x* = 0 (20)

e*st*+*i^x* = 0 (20)

(*x, z, t*) = *Ux* + *f*(*z*)e*st*+*i^x* (28)

= *B*2e*st*+*ix*+*^z z <* ⌘(*x, t*) (30)

= *B*2e*st*+*ix*+*^z z <* ⌘(*x, t*) (30)

velocity remains finite as \mathbf{v} **Transverse boundary conditions: velocity finite as** $\hspace{0.1cm} z \to \pm \infty$ *z* ! *±*1 (25) *z* ! *±*1 (25) **A22 Timite as** $z \to \pm \infty$

^f(*z*) + *^d*²*f*(*z*)

²

⇣

$$
A_2 = 0
$$

$$
B_1 = 0
$$

$$
\phi(x, z, t) = Ux + f(z)e^{st + i\kappa x}
$$

= Ux + A₁e^{st + i\kappa x - \kappa z}
= B₂e^{st + i\kappa x + \kappa z}
z < \eta(x, t)

e*st*+*i^x* = 0 (20)

$$
\phi(x, z, t) = Ux + f(z)e^{st + i\kappa x}
$$

= Ux + A₁e^{st + i\kappa x - \kappa z}
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z < \eta(x, t)

z ! *±*1 (25)

= *B*2e*st*+*ix*+*^z z <* ⌘(*x, t*) (30)

*A*² = 0 (26)

Interface boundary conditions will determine $\left\vert A_{1} \right\rangle, B_{2} \quad \& \quad s(\kappa)$

Two kinds of interface boundary condition:

- **- Kinematic BC: imposed by interface motion (moves with fluid),**
- **- Dynamic BC: imposed by interface dynamics (momentum/pressure balance)**

Kinematic constraint:

 $\frac{1}{2}$ **interface** city of f fli 2 <u>rticles at</u> **- Consider transverse velocity of fluid particles at interface** is positive to the control of the control

 $\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1}{\sqrt{1-\frac{1$

 (12)

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 \sim $(x, t) = \eta_0$ C
 $(z, t) = Ux + A_1 e^{st + i\kappa x - \kappa z}$ 1*,*2 = \mathbf{I} = $\frac{1}{\sqrt{2}}$ λ = $\frac{1}{2}$ \mathbf{r} . or $\eta(x,t) = \eta_0 e^{st + i\kappa x}$ $\phi(x, z, t) = Ux + A_1e^{st + i\kappa x - \kappa z}$ (*x, z, t*) = *B*2e*st*+*ix*+*^z* (35) **Displacement: Potential:**

al:
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$$
\phi(x, z, t) = Ux + A_1 e^{st + i\kappa x - \kappa z}
$$
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u(*t*) ⁼ *^A*^e or *s*
2 is positive to the set of the s
2 is positive to the set of @*z* @*t* @*x A*₂ *A***₂ ***A*₂ *A***₂ ***A*₃ *A*₃ *A***₃ ***A*₃ *A*₃ *A*₃ *A*₃ *A***₃ ***A*₃ *A*₃ *A*₃ *A***₃ ** Normal velocity of fluid particles must match that of interface

The general solution ⁱ^s the sum of normal modes If the real part of either *^s*¹ 2 ^A quick revision of differential equations Linear hydrodynamic stability theory ⁱ^s intimately tied to the properties of the linearised Navier-Stokes and Euler equations and their solutions. ^A clear understanding of the properties of differential equations (ordinary and partial), their solutions and the techniques used to obtain these ⁱ^s therefore indispensable if you wish to penetrate the rich and beautiful world of hydrodynamic stability. We here run briefly through some of the basics, and you are encouraged to revise these thoroughly before proceeding with the main material of these lectures. 1*st*-order ordinary differential equations gives, respectively The general solution is the sum of normal modes If the real part of either 2 A quick revision of differential equations Linear hydrodynamic stability theory is intimately tied to the properties of the linearised Navier-Stokes and Euler equations and their solutions. A clear under-*A*1e⌘ = (*^s* ⁺ *ⁱU*)⌘⁰ (38) Taylor series expansion of e⌘ = 1 ⌘ + *...* means that the kinematic boundary condition at the interface reduces to: @ @*z* = ⇣ @ @*t* ⁺ *^U* @ @*x* ⌘ ⌘ (33) *^A*1e⌘e*st*+*i^x* = (*^s* ⁺ *ⁱU*)⌘0e*st*+*i^x* (34) *^A*1e⌘ = (*^s* ⁺ *ⁱU*)⌘⁰ (35)

Equations of the form

Solution method

2.1

y^t

standing of the properties of differential equations (ordinary and partial), their solutions and the techniques used to obtain these is therefore indispensable if you wish to penetrate the rich and beautiful world of hydrodynamic stability. We here run briefly through some of the basics, and you are encouraged to revise these

thoroughly before proceeding with the main material of these lectures.

1*st*-order ordinary differential equations

(16)

Kinematic constraint:

tional

$$
-A_1\kappa e^{-\kappa\eta} = (s + i\kappa U)\eta_0
$$

⌘(*x, t*) = ⌘0e*st*+*i^x* (33)

Taylor series expansion of transverse structure
 A1estin 2018 *A*1e⌘e*st*+*i^x* = (*^s* ⁺ *ⁱU*)⌘0e*st*+*i^x* (37) @*z* @*t*

boundary condition at the interface reduces to:
The interface reduces to: the interface reduces to: the interface reduces to: the interface reduces to:

$$
e^{-\kappa\eta} = 1 - \kappa\eta + \dots
$$

Considering small disturbances the kinematic constraint reduces to:

$$
-A_1 \kappa = (s + i\kappa U)\eta_0
$$

$$
B_2 \kappa = s\eta_0
$$

Dynamic boundary condition at the interface: pressure must be continuous.

Bernoulli's equation holds on either side of the interface, where the fluid is irrota-

Dynamic constraint boundary condition at the interface reduces to: boundary condition at the interface reduces to:

- Pressure is continuous across the interface A1 $\overline{ }$ $\overline{ }$ us across the interf $\overline{\mathbf{C}}$

 (10)

- Bernoulli's equation holds on either side, but not at the interface **BALLO BELLET ENGINEEERS** holds on either side —
)
)
)
) \overline{e} but not at **1200 Example:**
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1200 Example:
1200 Example:
1200 Example:
1200 Example:
1200 Example:

(13)

 (1)

 (15)

 $f(x, t)$ $f(r, t)$ $\left(x, y, \frac{1}{\sqrt{2\pi}}\right)$ At $\eta^+(x,t)$

Taylor series expansion of e⌘ = 1 ⌘ + *...* means that the kinematic

4 Shear-flow instability
Dynamic constraint
- Pressure is continuous across the interface
- Bernoulli's equation holds on either side, but not at the interface
At $\eta^+(x,t)$
$\rho^{\partial \phi(x,z,t)} _{z=\eta^+} + \frac{\rho}{2} \left(\frac{\partial \phi(x,z,t) _{z=\eta^+}}{\partial x} \right)^2 + p(x,\eta^+,t) + \rho g \eta^+(x,t) = \text{constant}$
$= p_{\infty} + \frac{\rho U^2}{2}$
$\frac{U}{\sqrt{2}}$

At ⌘(*x, t*)

Unsteady Bernoulli equation *x*, bornoam $$ \overline{a} + $\$ Dynamic boundary condition at the interface: pressure must be continuous. ability bernoulling equation holds on either side of the interface, where the interface, where the fluid is in \overline{a}

$$
\rho \frac{\partial \phi(x, z, t)|_{z=\eta^+}}{\partial t} + \frac{\rho}{2} \left(\frac{\partial \phi(x, z, t)|_{z=\eta^+}}{\partial x} \right)^2 + p(x, \eta^+, t) + \rho g \eta^+(x, t) = p_{\infty} + \frac{\rho U^2}{2}
$$

*B*2 = *s*⌘⁰ (40)

Perturbation Ansatz: $\phi(x, z, t) = Ux + A_1 e^{st + i\kappa x - \kappa z}$ $\varphi(x, \lambda, v) = C x + Mc$

z=⌘⁺

boundary conditions at the interface reduces to: the interface reduces to: the interface reduces to: the inter
Interface reduces to: the interface reduces to: the interface reduces to: the interface reduces to: the interf

tional The pressure-momentum balancefield at ⌘⁺(*x, t*) is

$$
\left(\frac{\partial \phi(x, z, t)}{\partial x}\Big|_{z=\eta+}\right)^2 = U^2 + 2U i\kappa A_1 e^{-\kappa \eta^+} e^{st + i\kappa x} + \text{non-linear terms}
$$

$$
\left.\frac{\partial \phi(x, z, t)}{\partial t}\right|_{z=\eta+} = sA_1 e^{-\kappa \eta^+} e^{st + i\kappa x}
$$

$$
p(x, \eta^+, t) = p_{\infty} - \rho(s + i\kappa U)A_1 e^{-\kappa \eta^+} e^{st + i\kappa x} - \rho g \eta_0 e^{st + i\kappa x}
$$

$$
= p_{\infty} - \rho(s + i\kappa U)A_1 e^{st + i\kappa x} - \rho g \eta_0 e^{st + i\kappa x}
$$

4

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$$
p(x, \eta^+, t) = p_{\infty} - \rho(s + i\kappa U)A_1 e^{st + i\kappa x} - \rho g \eta_0 e^{st + i\kappa x}
$$

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t) =

is positive

is positive

$$
\eta^-(x,t)
$$

$$
\eta^-(x,t)
$$
\n
$$
p(x,\eta^-,t) = p_{\infty} - \rho s B_2 e^{st + ikx} - \rho g \eta_o e^{st + ikx}
$$
\nMatching across vortex sheet

\n
$$
sB_2 = (s + i\kappa U)A_1
$$
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Matching across vortex sheet

combined with the kinematic constraints above the kinematic constraints above the kinematic constraints above
The kinematic constraints above the kinematic constraints above the kinematic constraints above the kinematic

$$
= p_{\infty} - \rho s B_2 e^{st + ikx} - \rho g \eta_o e^{st + ikx}
$$

$$
sB_2 = (s + ikU)A_1
$$

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⁺ *^p*(*t*)*^y*

² (11)

*[±]*p²

1*,*2

Equations of the form

¹*,*

1*,*2

 $\mathcal{L}(\mathcal{L})$

t) =

or

1*,*2

²

²

linearised Navier-Stokes and Euler equations and their solutions. A clear under-

linearised Navier-Stokes and Euler equations and their solutions. A clear understanding of the properties of differential equations (ordinary and partial), their solutions and the techniques used to obtain these is therefore indispensable if you wish to penetrate the rich and beautiful world of hydrodynamic stability. We here run briefly through some of the basics, and you are encouraged to revise these

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1*st*-order ordinary differential equations

^e*st*+*i^x* ⇢*g*⌘0e*st*+*i^x* (46)

1*st*-order ordinary differential equations

² (11)

² (12)

Dynamic constraint Taylor series expansion of the kinematic series of \blacksquare matching across the vortex sheet gives:

Dynamic constraint
\n
$$
sB_2 = (s + i\kappa U)A_1
$$
\n**Recall kinematic constraint**

Recall kinematic constraint

dynamic constraint

\nscalar kinematic constraint

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$$
sB_2 = (s + ikU)A_1
$$
\necall kinematic constraint

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$$
-A_1\kappa = (s + ikU)\eta_0
$$
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$$
B_2\kappa = s\eta_0
$$
\ncombining constraints

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$$
s^2 + (s + ikU)^2 = 0
$$
\nINSTITUT PPRIME

Combining constraints

 $\overline{}$, whose roots who

4 Shear-flow instability	
U	
Dynamic constraint	\n $sB_2 = (s + i\kappa U)A_1$ \n
Recall kinematic constraint	\n $-A_1\kappa = (s + i\kappa U)\eta_0$ \n
Combining constraints	\n $s^2 + (s + i\kappa U)^2 = 0$ \n

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*s*2

combine with the kinematic constraints above the kinematic constraints above the kinematic constraints above t
The kinematic constraints above the kinematic constraints above the kinematic constraints above the kinematic

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$$
s^2 + (s + i\kappa U)^2 = 0
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⌘0*/* = (*s* + *iU*)

 (1)

 (1)

 (12)

 (12)

$$
s=-\frac{1}{2}i\kappa U\pm\frac{1}{2}\kappa U
$$

$$
\eta(x,t) = \eta_0 e^{\frac{1}{2}\kappa U t + i\kappa (x - \frac{1}{2}Ut)}
$$

⌘*/* (50)

1*st*-order ordinary differential equations

I N S T I T U T P P R I M E

Instable mode travelling in the *x*-direction

sponentially with growth rate $kU/2$

certain range of wavenumber will be unstable

considered by letting $s=i\omega$ in
 $(s+i\kappa U)^2=0$ Linear hydrodynamic stability theory is interested to the properties of the pr linearised Navier-Stokes and Euler equations and their solutions. A clear understanding of the properties of differential equations (ordinary and partial), their solutions and the techniques used to obtain Linear hydrodynamic stability theory is interested to the properties of the pr linearised Navier-Stokes and Euler equations and their solutions. A clear understanding of the properties of differential equations (or differential), the participation of the participation solutions and the techniques used to obtain the techniques used to obtai *u* = *k* = *A*
*u***(***t***)** *u***)** *u* **=** *A***e2***t* **(***t***)** *u* **=** *A***e2***y* **(***n***)** *u* **=** *A***e2***y* **(***n***)** *u* **=** *B* **For each wavenumber,** *k***, there is an unstable mode travelling in the** *x***-direction With phase speed** *U/2* **and growing exponentially with growth rate** *kU/2* matching across the vortex sheet gives: where α *s* which the specifical provision of $\frac{1}{2}$

⁺ *^p*(*t*)*^y* $\overline{}$ to penaltrate the rich and beautiful world of $\overline{}$ run briefly through some of the basic solu \blacksquare run briefly through some of the basics, and you are encouraged to revise these In the case of finite thickness only a certain range of wavenumber will be unstable
The spatial stability problem can be considered by letting ϵ_{min} in **In the case of finite thickness only a certain range of wavenumber will be unstable**

considered by letting $s=i\omega$ in
+ $(s+i\kappa U)^2=0$
INSTITUT PPRIME **The spatial stability problem can be considered by letting** $s=iv$ **in** *s*2 ⌘0*/* = (*s* + *iU*) 2 $3 - 100$

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$$
s^{2} + (s + i\kappa U)^{2} = 0
$$

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1*st*-order ordinary differential equations

² (11)

² (12)

1*st*-order ordinary differential equations

thoroughly before proceeding with the main material of these lectures.

thoroughly before proceeding with the main material of these lectures.

4 Shear-flow instability solution 3 and 3 shear-flow instability sequences the shear-layer. *^p*(*x,* ⌘*, t*) = *^p*¹ ⇢*sB*2e*st*+*i^x* ⇢*g*⌘0e*st*+*i^x* (48)

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Spatial problem

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s^{2} + (s + ikU)^{2} = 0
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\nSpatial problem

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U^{2}\kappa + 2\omega U\kappa + \omega^{2} = 0
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\nRoots

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\omega \quad \omega
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Spatial problem

$$
s^{2} + (s + i\kappa U)^{2} = 0
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Spatial problem

$$
U^{2}\kappa + 2\omega U\kappa + \omega^{2} = 0
$$

Roots

$$
\kappa = -\frac{\omega}{n} + i\frac{\omega}{n}
$$

Roots

$$
s^{2} + (s + ikU)^{2} = 0
$$

Spatial problem

$$
U^{2}\kappa + 2\omega U\kappa + \omega^{2} = 0
$$

Roots

$$
\kappa = -\frac{\omega}{U} \pm i\frac{\omega}{U}
$$

Write down space-time behaviour of the spatial instability.

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space-time behaviour of the spatiant

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T $U=U$
me behaviour of the spatial instability.
INSTITUT PPRIME Write down space-time behaviour of the spatial instability.

THE REVIS TITUT PPRIME **Write down space-time behaviour of the spatial instability.**

Solution method

²

Linear hydrodynamic stability theory is intimately tied to the properties of the linearised Navier-Stokes and Euler equations and their solutions. A clear understanding of the properties of differential equations (ordinary and partial), their solutions and the techniques used to obtain these is therefore indispensable if you wish to penetrate the rich and beautiful world of hydrodynamic stability. We here run briefly through some of the basics, and you are encouraged to revise these

Linear hydrodynamic stability theory is intimately tied to the properties of the linearised Navier-Stokes and Euler equations and their solutions. A clear understanding of the properties of differential equations (ordinary and partial), their solutions and the techniques used to obtain these is therefore indispensable if you wish to penetrate the rich and beautiful world of hydrodynamic stability. We here run briefly through some of the basics, and you are encouraged to revise these

 $\overline{}$

1*st*-order ordinary differential equations

² (11)

² (12)

1*st*-order ordinary differential equations

thoroughly before proceeding with the main material of these lectures.

thoroughly before proceeding with the main material of these lectures.

Résumé

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@*z*² = 0 (15)

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لِيَّ الْمَسْرِينِ لَّذَا يَا الْمَجْمَعِينَ بِمَا يَا الْمَجْمَعِينَ بِهِ الْمَجْمَعِينَ بِهِ مِنْ الْمَجْمَع
Potential flow assumed above and below the vortex sheet: Laplace's equation If the real part of either *^s*¹ $\mathbf{place}^{\prime}\mathbf{s} \mathbf{ equation}$ gives, respectively. c respectively. 1*,*² = *± i ow the vortex sheet: Laplace's equation* $\overline{\text{th}}$ \cdot orte brtex sneet: Lapiace's equation ⌘0*/* = (*s* + *iU*)

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\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0
$$

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2020 **Disturbance Ansatz: velocity potential with normal modes:**
 Disturbance Ansatz: velocity potential with normal modes: z: velocity potential with normal modes:

$$
\phi(x, z, t) = Ux + f(z)e^{st + i\kappa x}
$$

Leads to ODE for transverse structure: الاه.
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$$
\varphi(x, z, v) = c \, x + f(z) \mathbf{1}
$$
\nis also to ODE for transverse structure:

\n
$$
-\kappa^2 f(z) + \frac{d^2 f(z)}{dz^2} = 0
$$
\n
$$
f(z) = A_1 e^{-\kappa z} + A_2 e^{\kappa z}
$$

@*^z* @*^w*

@*x* :

2 **General solution:**

$$
dz^{2}
$$

eneral solution: $f(z) = A_1 e^{-\kappa z} + A_2 e^{\kappa z}$
 $f(z) = B_1 e^{-\kappa z} + B_2 e^{\kappa z}$

 \overline{c} **Boundary and interface matching conditions:**

boundary and interface matching conditions:

\n
$$
\eta(x,t) = \eta_0 e^{\frac{1}{2}\kappa U t + i\kappa (x - \frac{1}{2}U t)}
$$
\nINSTITUT PPRIMF

Can assess stability of system by considering energy before and after introduction of a perturbation

Alternatively one can consider linear dynamics of problem arnes or problem

For vortex-sheet problem:

- **Simplification of governing equations: Laplace & Bernoulli** Form of district the distribution of the process
- Introduction of normal modes in *x* and $t \qquad \phi(x, z, t) = Ux + f(z)e^{st + i\kappa x}$
- **Solution for** *s(k)*
- ² ⁼ ² *f*(*z*)e*st*+*i^x* (19) *d*² *d*²*f*(*z*) @*x*² **- Transverse structure can also be determined (eigenfunction) which means that the entire space-time structure is obtained**

⇣ cepuon raufer u
e same: reduce *dz*² .
n an ruie: numericai
DE system to ODE system *i* nu \mathbf{n} Analytical solution possible for vortex sheet: exception rather than rule: numerical *d*² **solution is usually necessary, but rationale is the same: reduce PDE system to ODE system that takes form of an eigenvalue problem**

 A second equation is required for a complete description of the flow. One possibility \mathcal{A}