INSTITUT PPRIME
CNRS–UPR–3346 • UNIVERSITÉ DE POITIERS • ENSMA

DÉPARTEMENT D2 – FLUIDES
THERMIQUE ET COMBUSTION

An introduction to hydrodynamic stability

Lecture 1: General introduction

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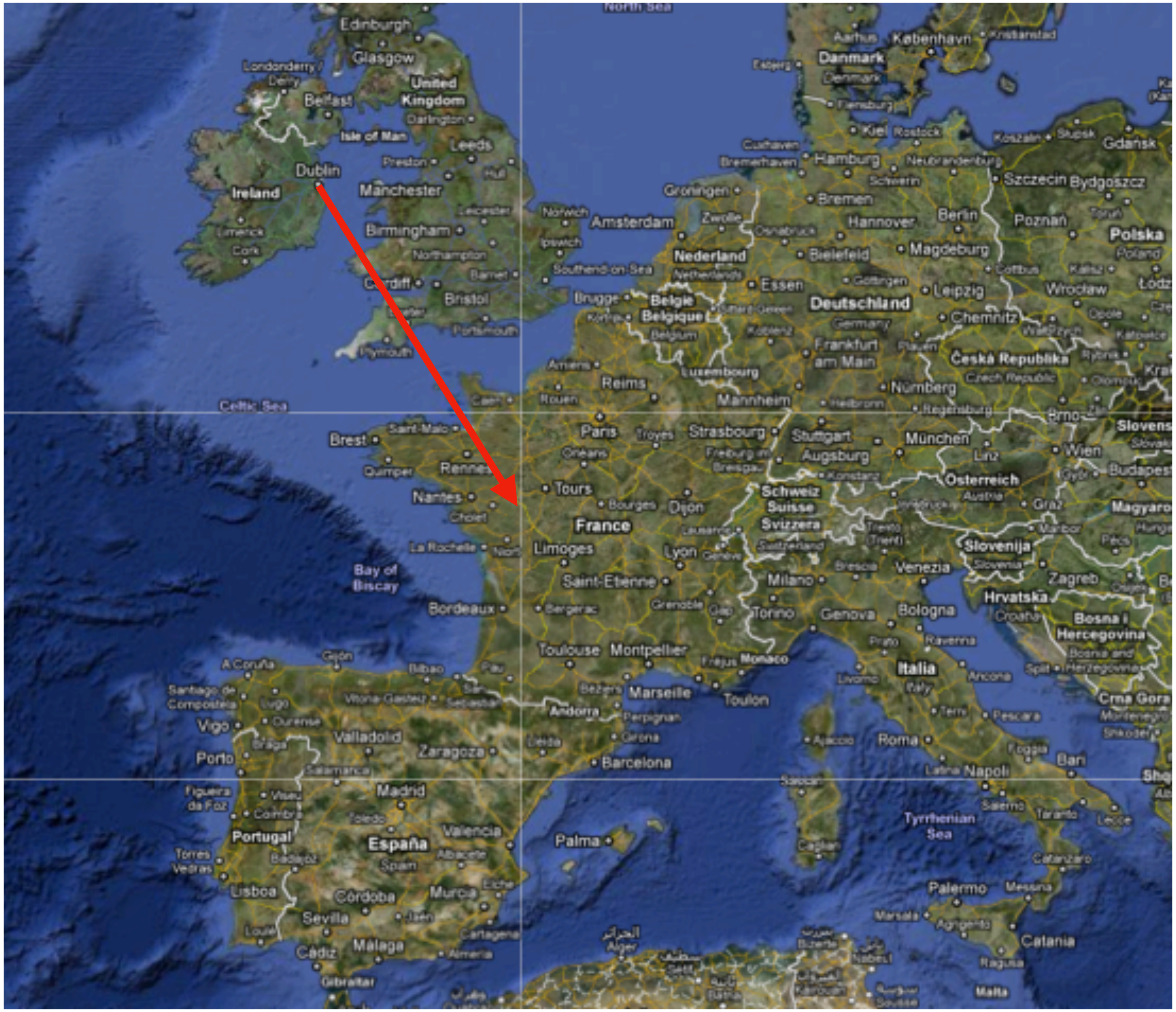
Who am I? Where do I come from?

L'imposante distillerie Jameson , Midleton, Comté de Cork

Le centre "The Jameson Heritage" à Midleton, Comté de Cork

Who am I? Where do I come from?



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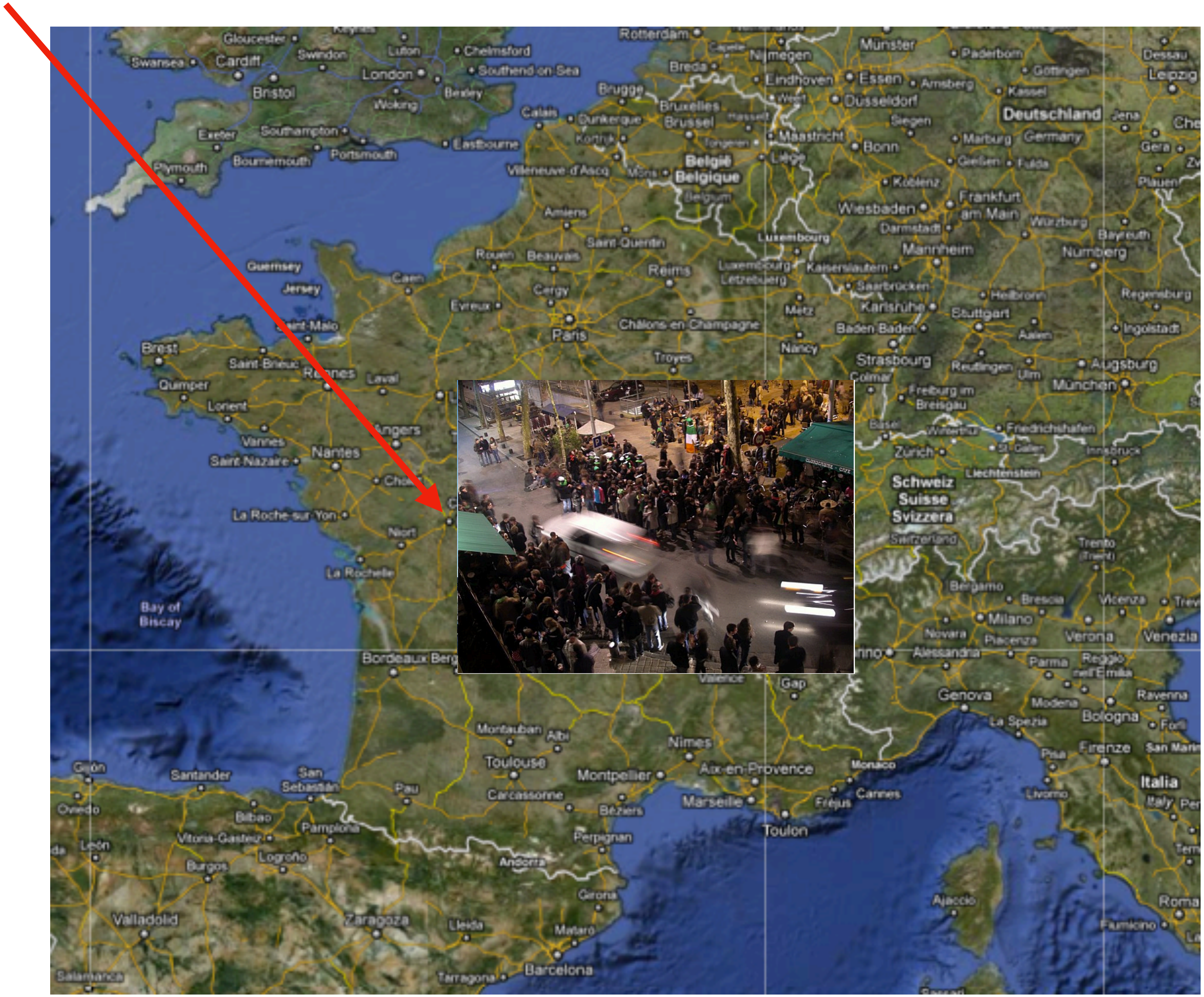

Lecture 1: General introduction

- Overview of different fluid instabilities
- Basic notions of stability
- Energy approach versus direct consideration of linear dynamics
- Shear-flow (Kelvin-Helmholtz instability)

Lecture 2: The governing equations for fluid instability

- Rayleigh equation
- Orr-Sommerfeld equation
- The Squire transformation

Lecture 3: Numerical methods

- Solving eigenvalue problems
- Inviscid temporal instability of a 2D mixing layer

Lecture 4: Viscous instability

- Viscosity and stability of plane Poiseuille flow
- Orr-Sommerfeld solution for mixing layer
- Boundary-layer instability
- Rayleigh's inflection-point theorem

Lecture 5: The spatial and spatiotemporal stability problems

- The linearised equations in full form
- The 2D mixing layer
- The compressible round jet
- Spatiotemporal stability
- Non-parallel flows

Lecture 6: Non-modal instability

- The enigma of pipe flow
- The Orr-Sommerfeld-Squire system
- The initial-value problem
- Non-normality and transient growth

Lecture 7: Beyond the critical point

- Rayleigh-Bénard convection
- State-space representation of dynamics systems
- Local bifurcation theory

Lecture 8: Weakly non-linear stability

- The Eckhaus equation
- The Stuart-Landau equation
- The Ginzburg-Landau equation

Lecture 9: Linear stability of fully turbulent flows

- Wavepackets and turbulent jet noise*

*Jordan & Colonius (2013) Ann. Rev. Fluid Mech.

1. Some videos to kick off

- Broad range of instability phenomena
- Importance applications

2. Why study hydrodynamic stability?

3. Basic notions of stability

- Stable and unstable systems
- Energy consideration
- Linear dynamics

4. Shear-flow (Kelvin-Helmholtz) instability

1. Some videos to kick off

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i. Kelvin-Helmholtz shear-flow instability

ii. Poisueille flow - Reynolds experiment

iii. Rayleigh-Plateau capillary instability

iv. Taylor-Couette centrifugal instability

v. Rayleigh-Bénard convective instability

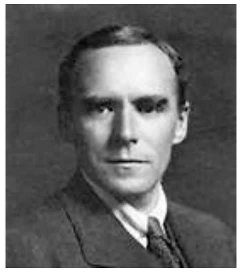
vi. Rayleigh-Taylor interface instability (in stratified fluid)

vii. Tollmien-Schlichting viscous instability in wall-bounded flow

viii. Von-Karman wake instability

Rayleigh

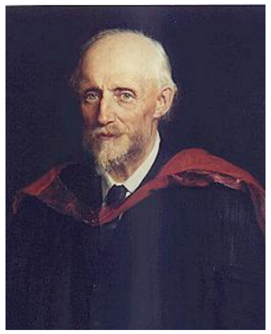
Taylor



Kelvin

Helmholtz

Reynolds



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Couette

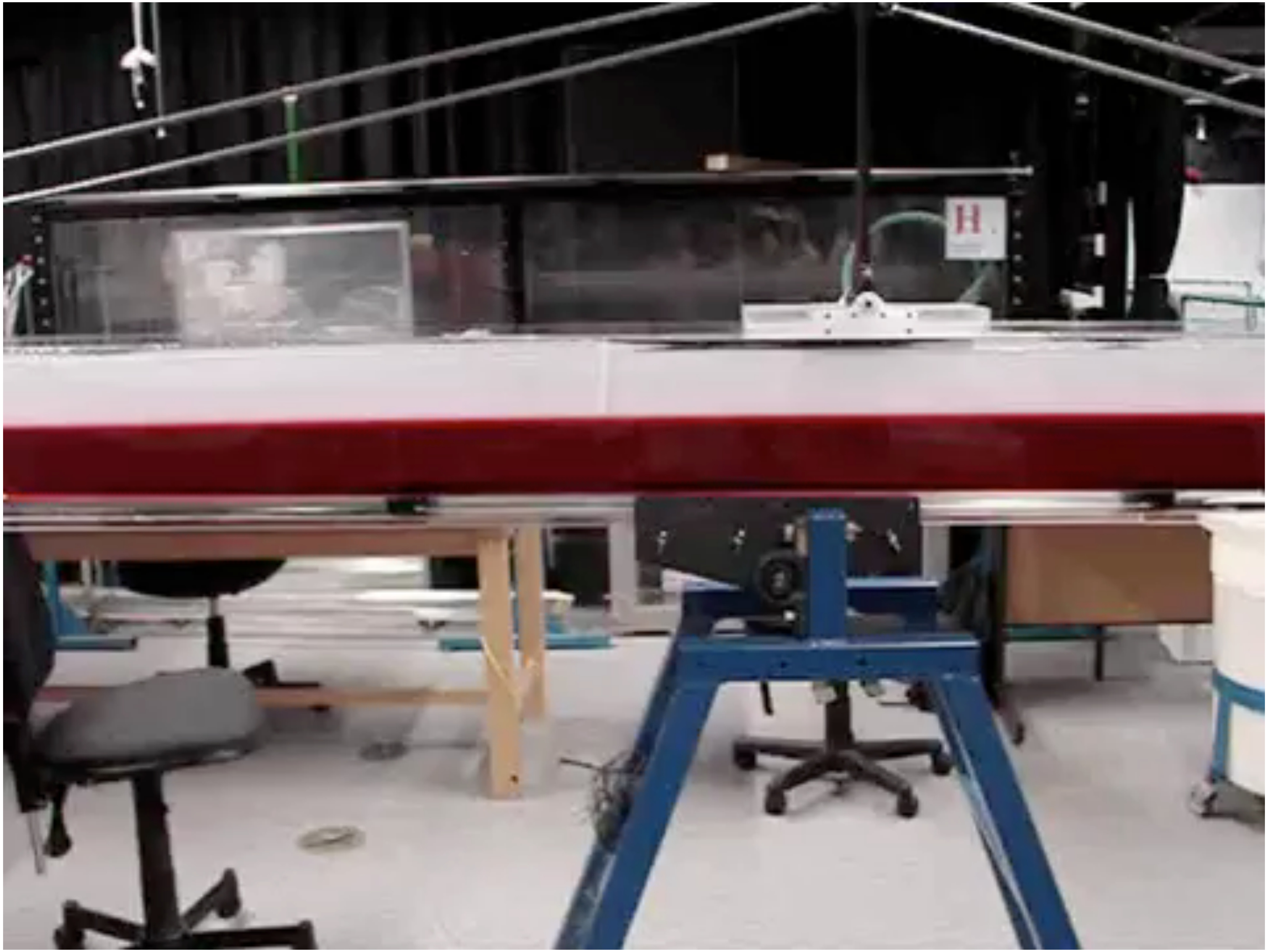
Plateau

Sommerfeld

Von Karman

Kelvin-Helmholtz
shear-flow instability

Kelvin-Helmholtz shear-flow instability

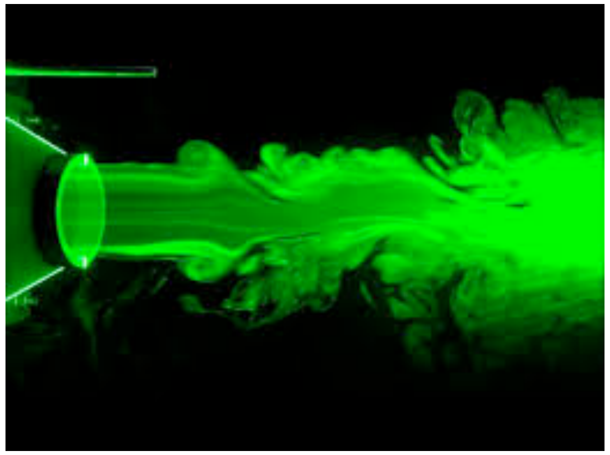


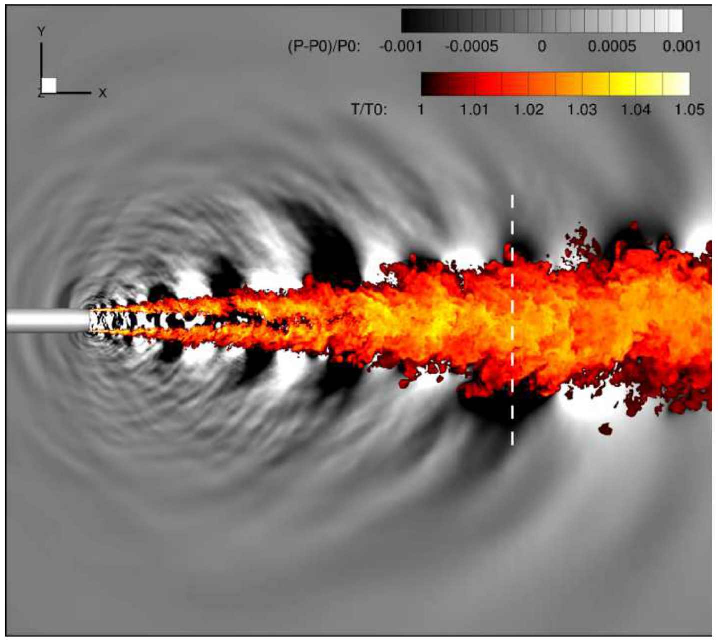
Underpins transition in all inflectional shear-flows

**Jets, wakes, shear-layers, jets in cross flow,
separation from bluff bodies, separation from airfoil,…**

Underpins coherent structures in fully turbulent inflectional shear flows

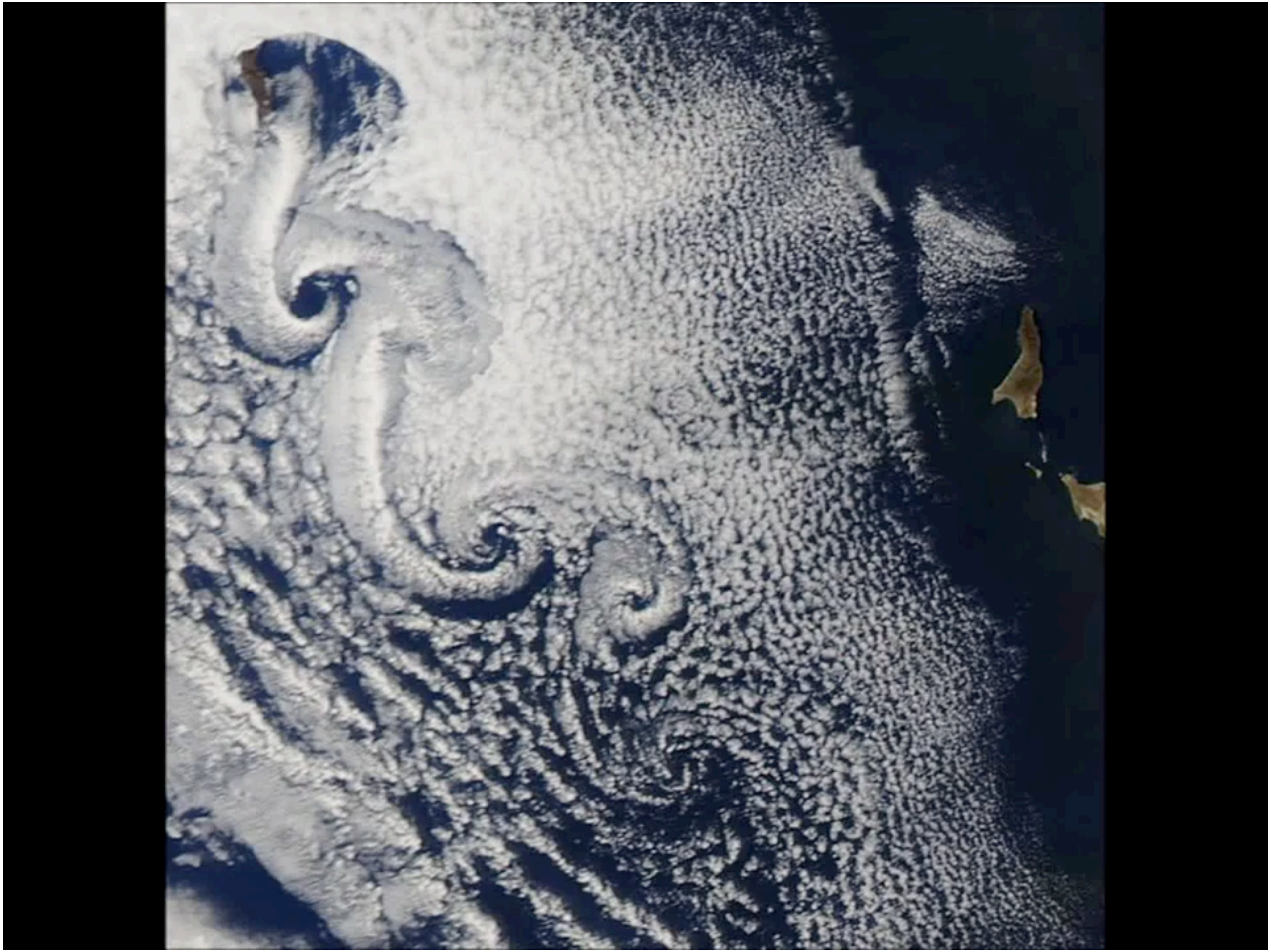
Important in aerodynamically generated noise (jet noise).





Von-Karman
wake instability

Von-Karmen wake instability



Von-Karman wake instability

Underpins drag induced by bluff bodies, landing gear of aircraft, for example

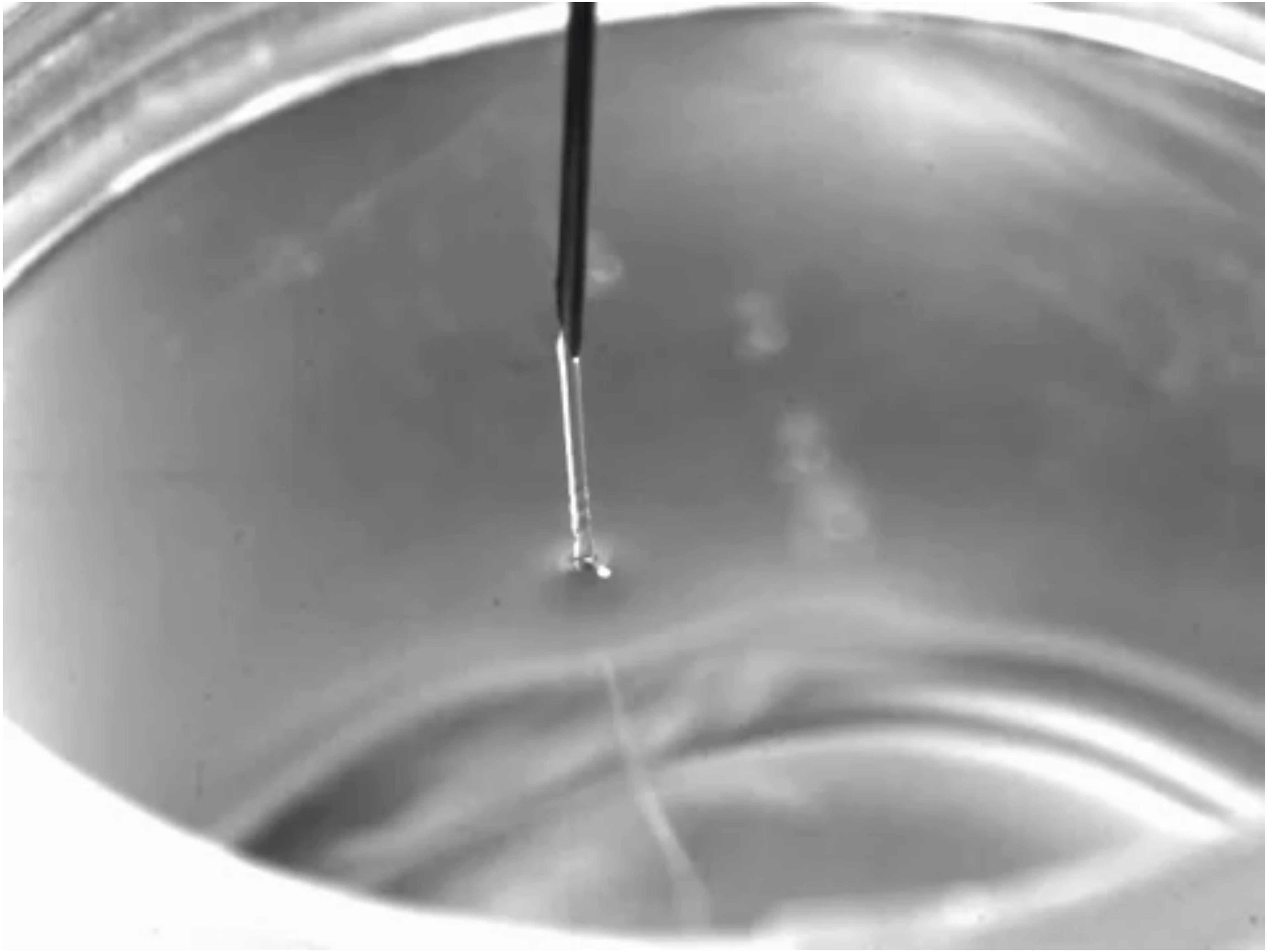
Unsteady loads on offshore structures

Resonance in heat exchangers

Important in aerodynamically generated noise - landing-gear noise

Rayleigh-Plateau
capillary instability

Plateau-Rayleigh capillary instability



Flow from a tap

Ink-jet printers

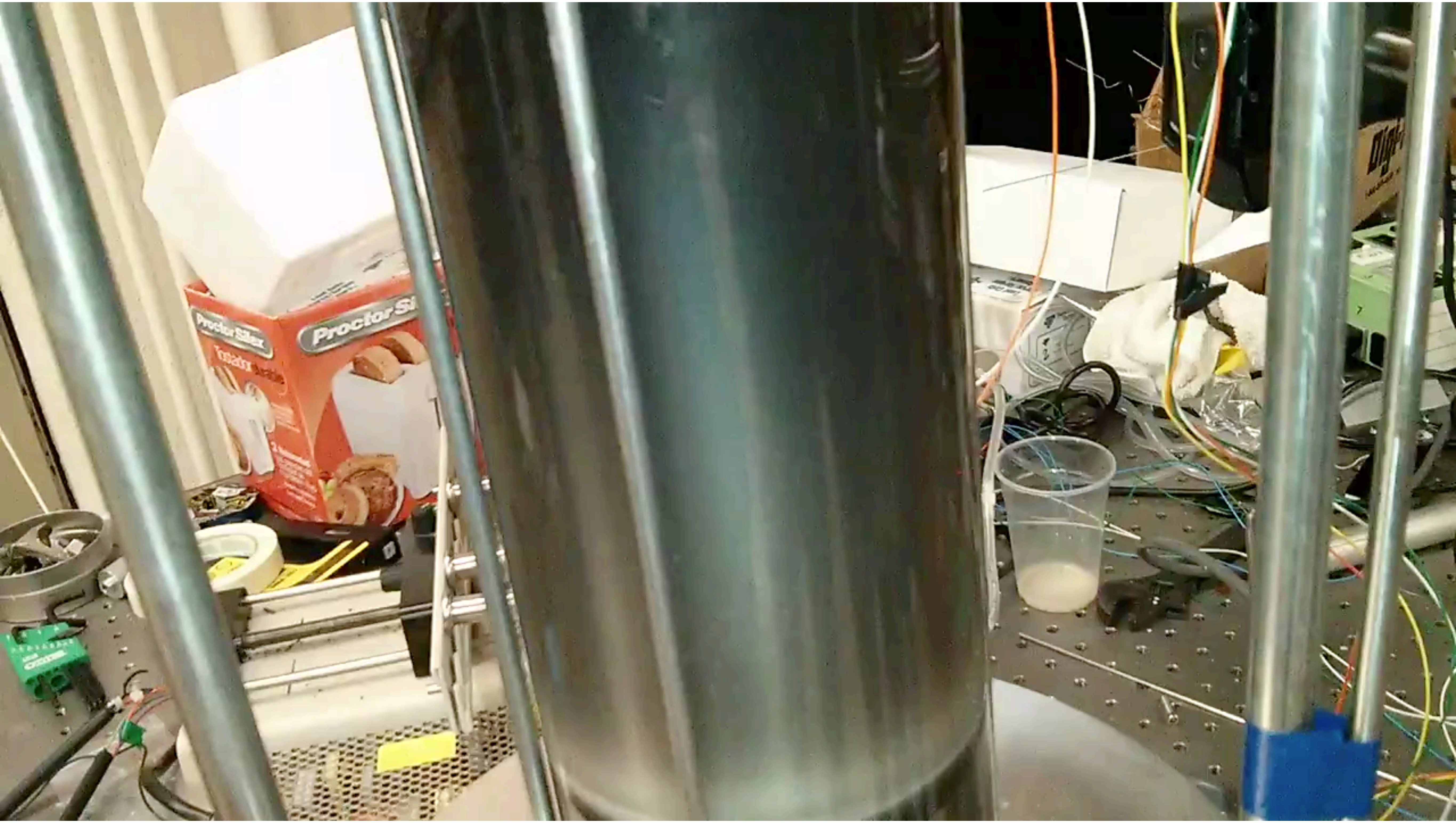
- Promote drop formation
  - Droplets charged electrically
  - Guided using electric field
- Essential that droplets have the same mass

Spinning of nylon

- Important to prevent droplet formation to obtain constant-diameter fibres

Fluid-injection systems: atomisation important for combustion efficiency

Taylor-Couette instability



Lubrification in journal bearings

Rotating filtration: extracting plasma from blood, water purification,…

Magnetohydrodynamics: magnetic fields of planets

Rotational viscometers

Rayleigh-Bénard instability

Rayleigh-Bénard instability

Astrophysics - heat/energy transfer in outer atmosphere of stars

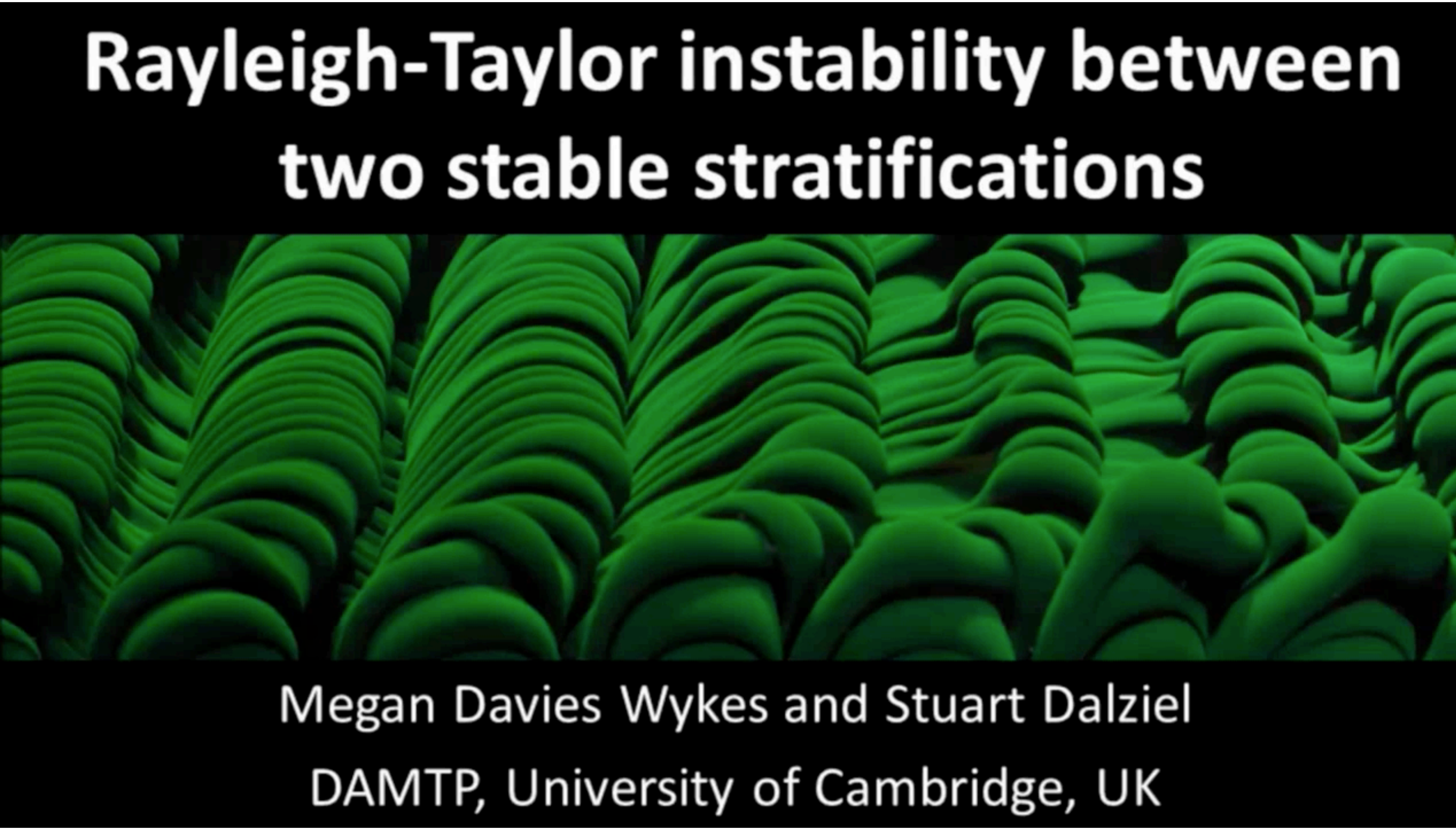
Geophysics - movement of Earth's mantle

Atmospheric science: weather systems, including long terms effects (ice-ages)

Solar energy systems

Nuclear systems

Material processing


Rayleigh-Taylor instability between two stable stratifications
Megan Davies Wykes and Stuart Dalziel
DAMTP, University of Cambridge, UK

Heat transfer

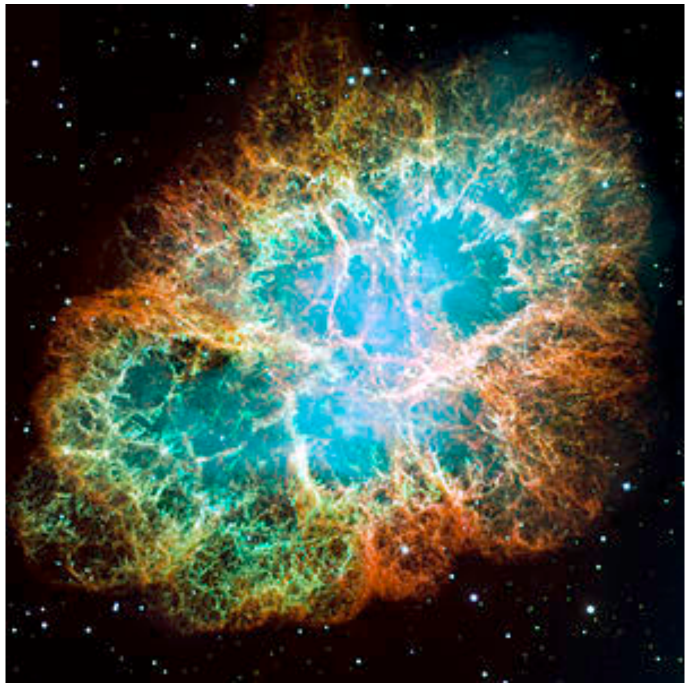
Inertial confinement fusion

Nuclear bombs

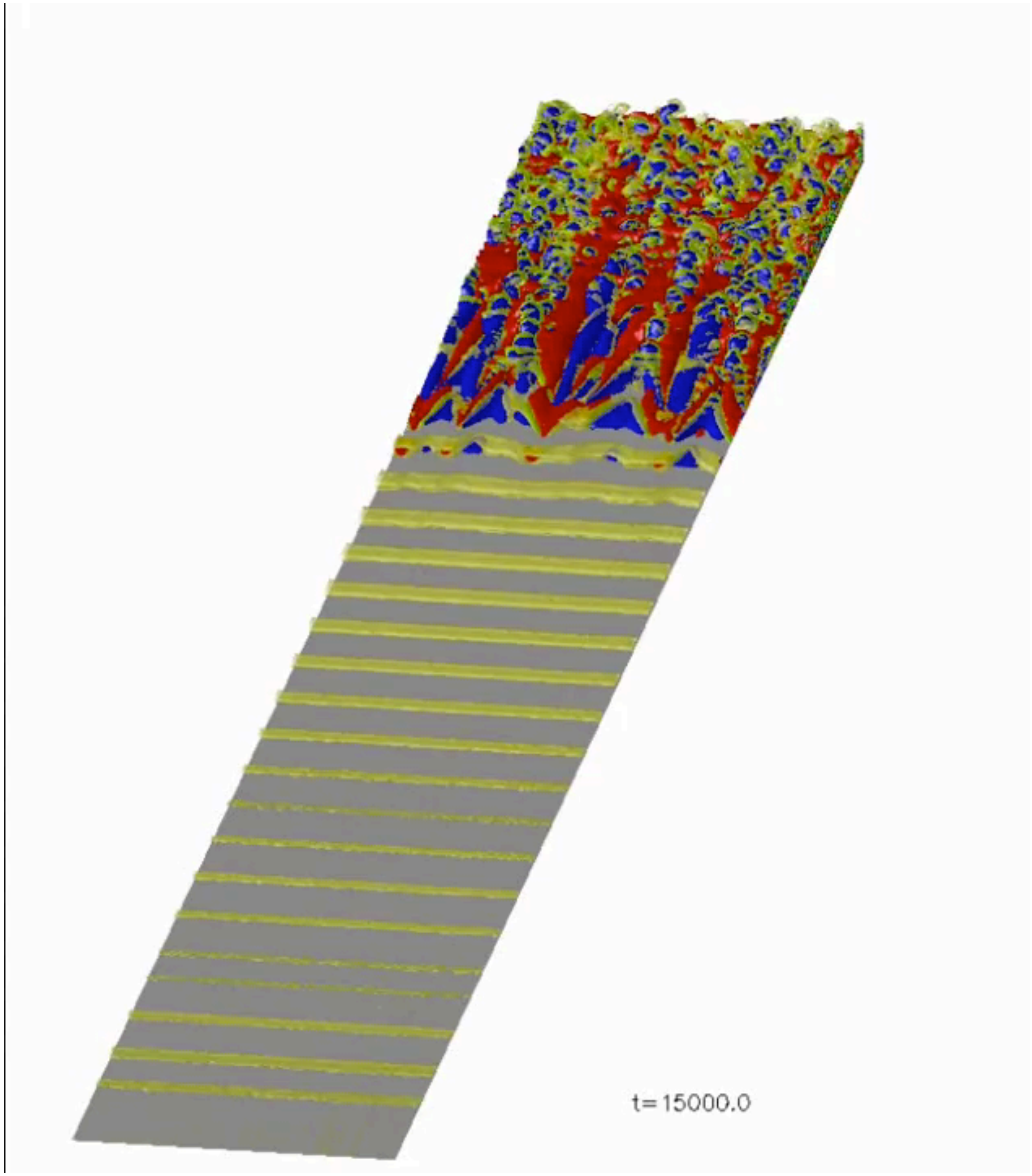
Supernova explosions

Solar coronas

Lava lamp



Tollmien-Schlichting instability



Transition of boundary layers

Drag reduction on wings - laminar-wing projects

2. Why study hydrodynamic stability?

2. Why study hydrodynamic stability?

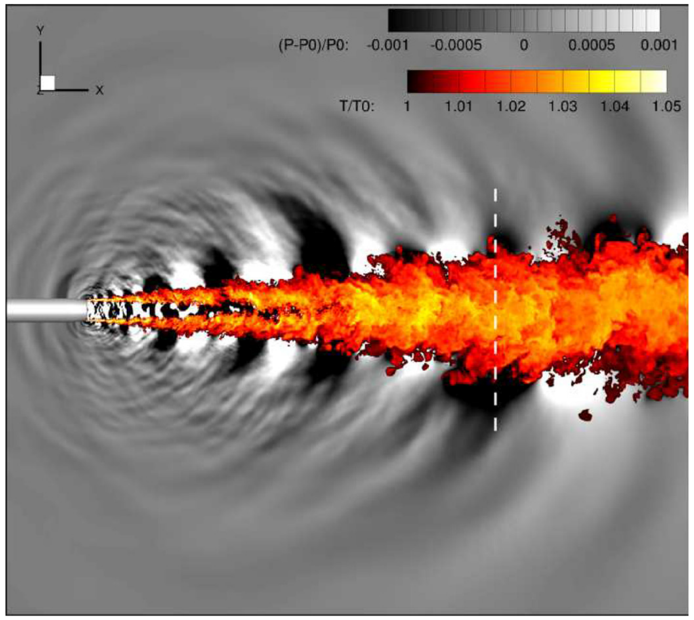
In fluid mechanics, clear understanding is the exception rather than the rule

Turbulence is poorly understood due to nature of governing equations:

- non-linear, in 4 dimensions, 6 dependent variables,…
- We can obtain approximate solutions using (very) large computers
- but solution does not imply understanding

Hydrodynamic stability theory & analysis:

- Direct and relatively complete understanding is available:
  - linearity, analytical solutions
- The magic of fluid mechanics is here most accessible:
  - the videos are visually striking
  - understanding makes them even better
- Surprisingly pertinent in many fully turbulent flows…



3. Some basic notions of stability

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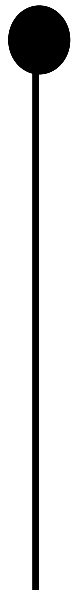
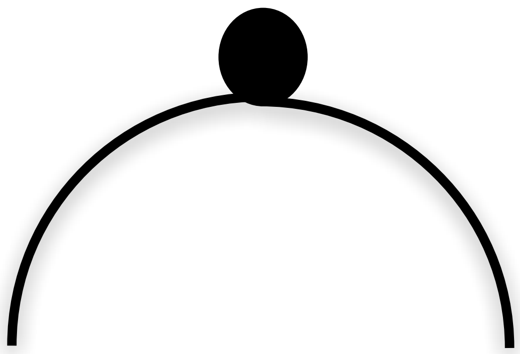
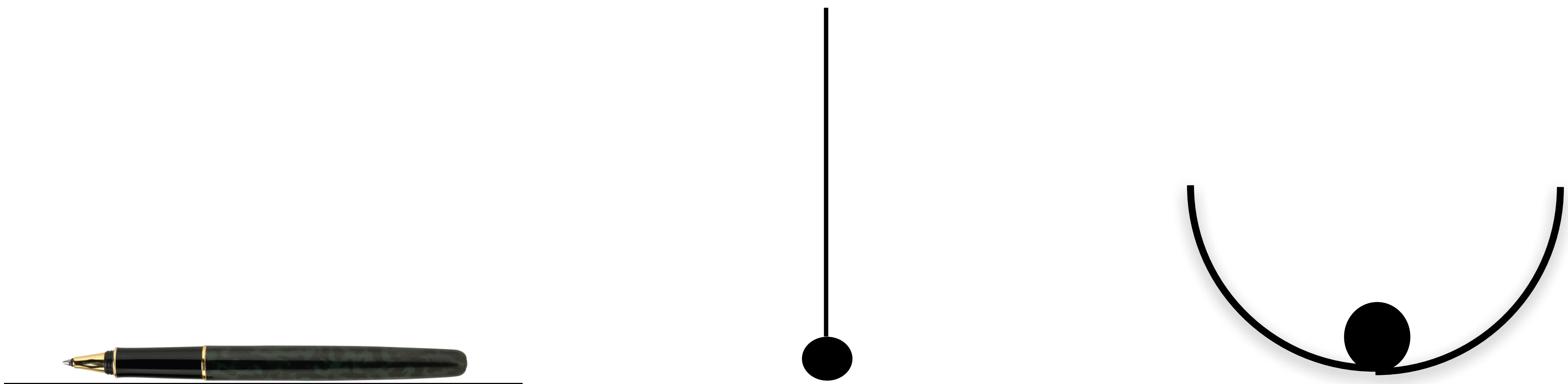
Unstable systems

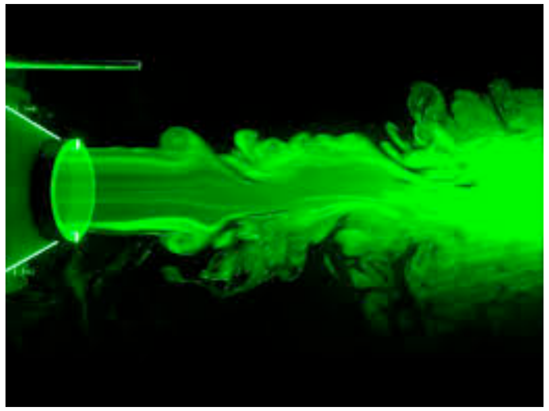




Stable systems



What have those systems got to do with fluid mechanics?



We can ask of a fluid system, be it still or in motion:

- What will be its response to an infinitesimally small perturbation?

We can consider the problem in terms of:

- Energy states
- Linear dynamics

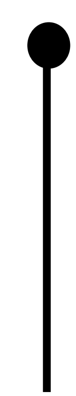
3. Some basic notions of stability

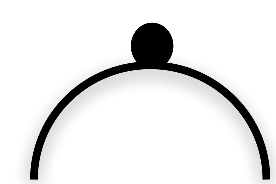
Questions typically asked in a stability analysis:

1. Is system **STABLE** or **UNSTABLE**?

2. How does this state change as some parameter is changed?
   - typically the Reynolds number in shear-flow problems

3. How will system behave in response to a small perturbation?
   - What are its **DYNAMICS**?





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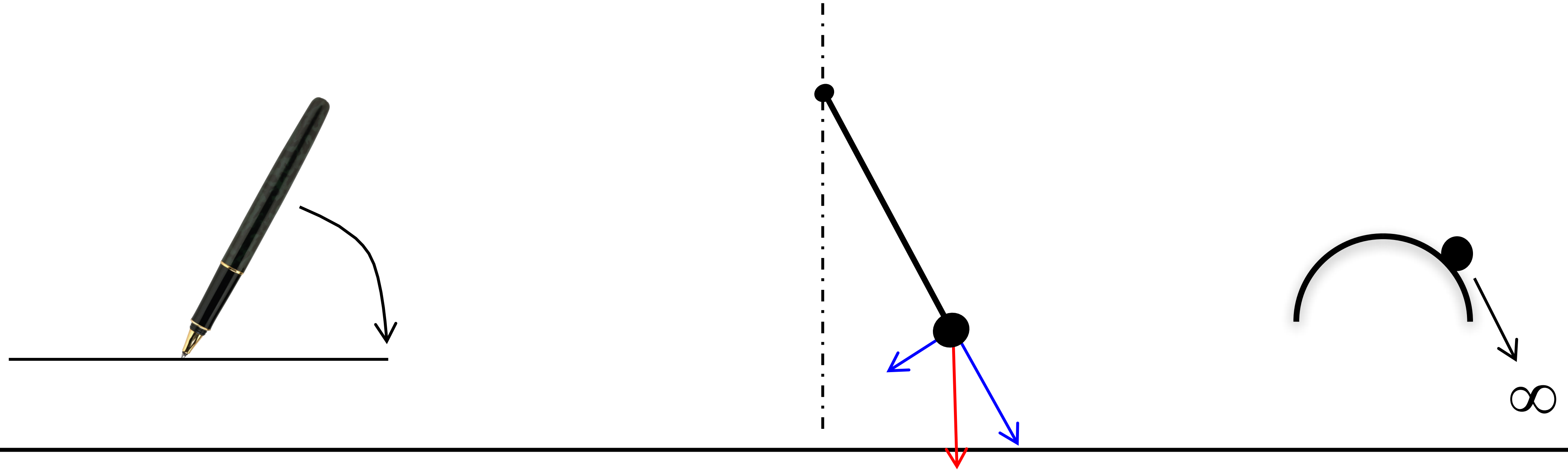


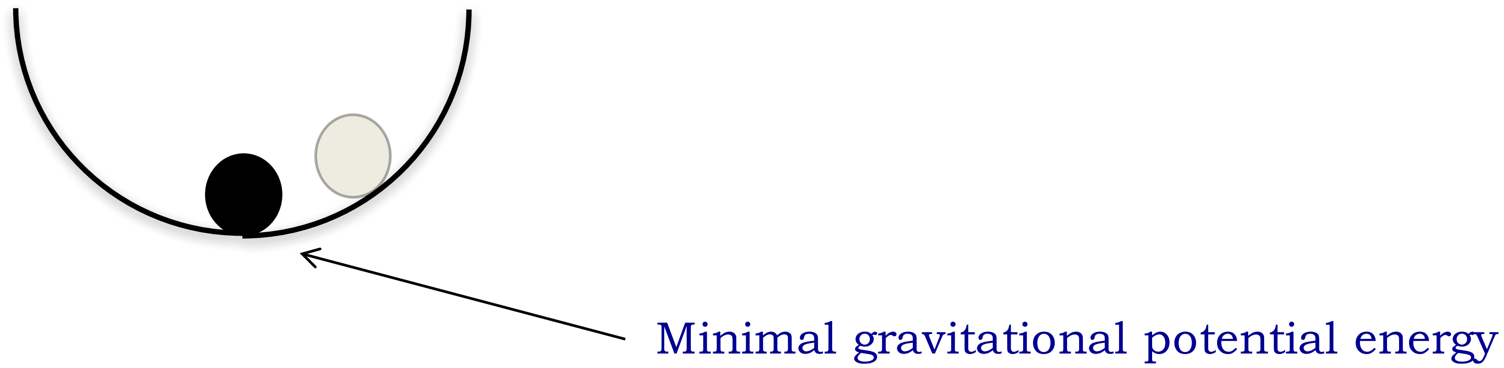
Energy consideration

Energy consideration

If perturbed system has MORE energy than steady state:

- **Energy required to maintain perturbation**
- **System is STABLE**

Stability corresponds to a state of minimal energy

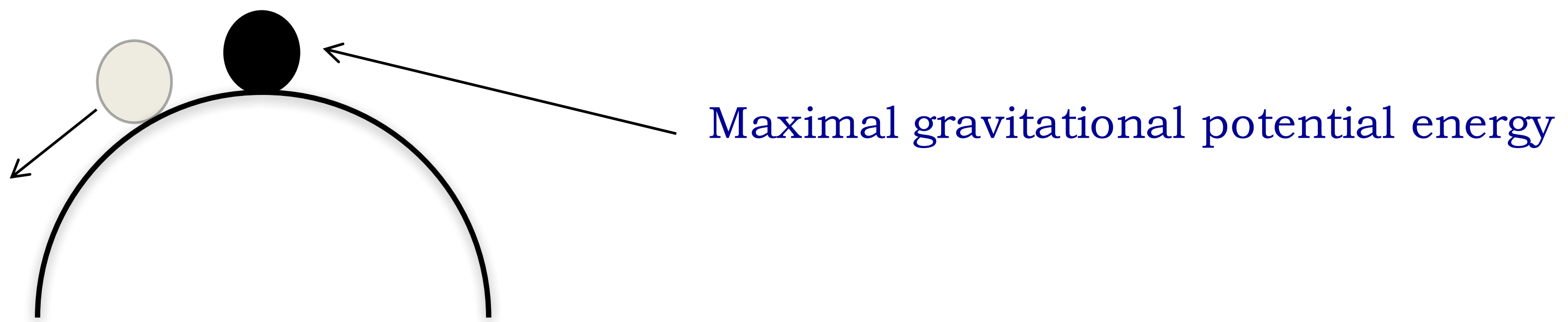


Energy consideration

If perturbed system has LESS energy than steady state:

- No external energy required to amplify perturbation
- System is UNSTABLE

An unstable system has maximal energy and will release this in response to an infinitesimal perturbation so as to incline to a lower energy state.



Consideration of linear dynamics

Linearised equations of motion for the system, which could be:

- a pendulum,
- a bouncing ball,
- a rocking boat,
- a flowing fluid,
- ...

Dynamics governed by ODE

$$\frac{du}{dt} = \mathcal{L}(u)$$
$$\frac{du}{dt} = \lambda u$$

Solution

$$u(t) = A\mathrm{e}^{\lambda t}$$

3. Some basic notions of stability

ODE

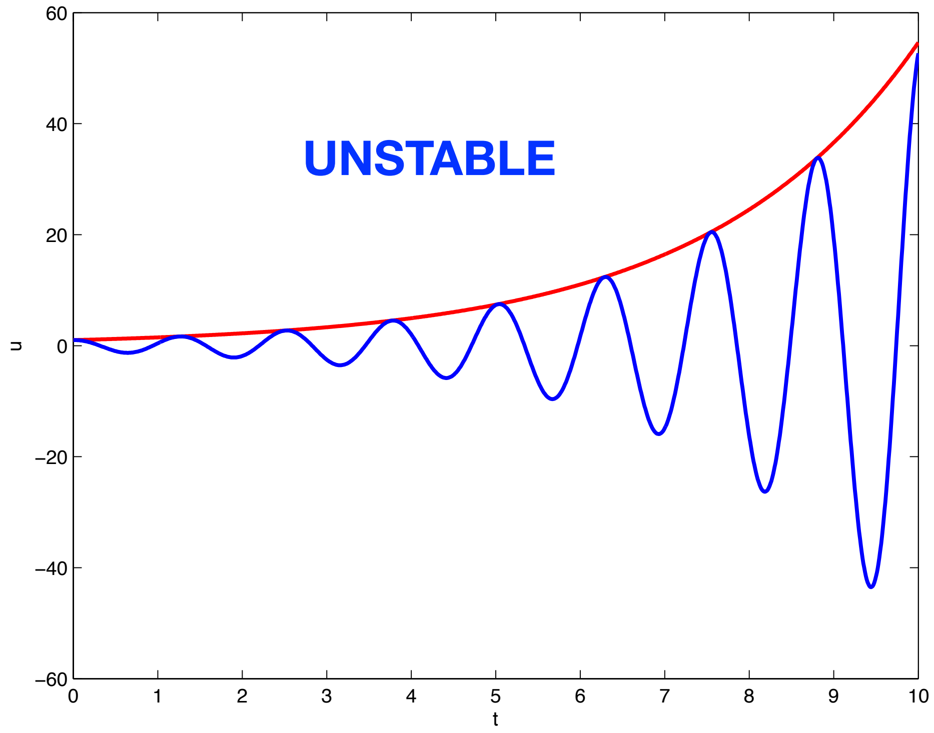
$$\frac{du}{dt} = \lambda u$$

Solution

$$u(t) = Ae^{\lambda t}$$

Sketch the dynamics that correspond to the following scenarios:

$$\lambda = a + ib$$
$$\lambda = -a + ib$$
$$\lambda = a + i0$$
$$\lambda = -a + i0$$



Consider a spring-mass-damper system, governed by

$$\frac{d^2u}{dt^2} + 2\beta\frac{du}{dt} + \gamma u = 0$$

Perturbations $u(t)$ proportional to $\mathrm{e}^{\lambda t}$

Obtain characteristic equation: $\lambda^2 + 2\beta\lambda + \gamma = 0$

$$\beta^2 > \gamma \qquad \lambda_{1,2} = -\beta \pm \sqrt{\beta^2 - \gamma}$$

$$\beta^2 < \gamma \qquad \lambda_{1,2} = -\beta \pm i\sqrt{\gamma - \beta^2}$$

General solution $u(t) = A\mathrm{e}^{\lambda_1 t} + B\mathrm{e}^{\lambda_2 t}$

ODE

$$\frac{d^2u}{dt^2} + 2\beta\frac{du}{dt} + \gamma u = 0$$

Roots

$$\lambda_{1,2} = -\beta \pm \sqrt{\beta^2 - \gamma}$$

$$\lambda_{1,2} = -\beta \pm i\sqrt{\gamma - \beta^2}$$

Solution

$$u(t) = A\mathrm{e}^{\lambda_1 t} + B\mathrm{e}^{\lambda_2 t}$$

System is UNSTABLE if at least one root has positive real part

This is the case if either $\beta < 0$ **and/or** $\gamma < 0$

ODE

$$\frac{du}{dt} = \lambda u$$

Solution

$$u(t) = A\mathrm{e}^{\lambda t}$$

ODE

$$\frac{d^2u}{dt^2} + 2\beta\frac{du}{dt} + \gamma u = 0$$

Solution

$$u(t) = A\mathrm{e}^{\lambda_1 t} + B\mathrm{e}^{\lambda_2 t}$$

What is the connection with fluid dynamics?

Partial Differential Equations → **Solutions comprise a greater wealth of phenomena**

Summary

Stability of a system can be assessed by:

1. Considering the energy change when system is perturbed

$\Delta E > 0$ → Energy input required to maintain perturbation, system is **STABLE**

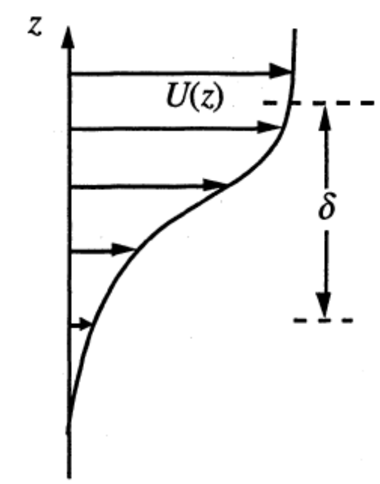
$\Delta E < 0$ → Energy released from system when perturbed, system is **UNSTABLE**

2. Considering the linear dynamics, solutions $\mathrm{e}^{\lambda t}$

$\mathrm{Re}\{\lambda\} < 0$ → Linear perturbations decay exponentiall, system is **STABLE**

$\mathrm{Re}\{\lambda\} > 0$ → Linear perturbations grow exponentiall, system is **UNSTABLE**

4. Kelvin-Helmholtz shear-flow instability

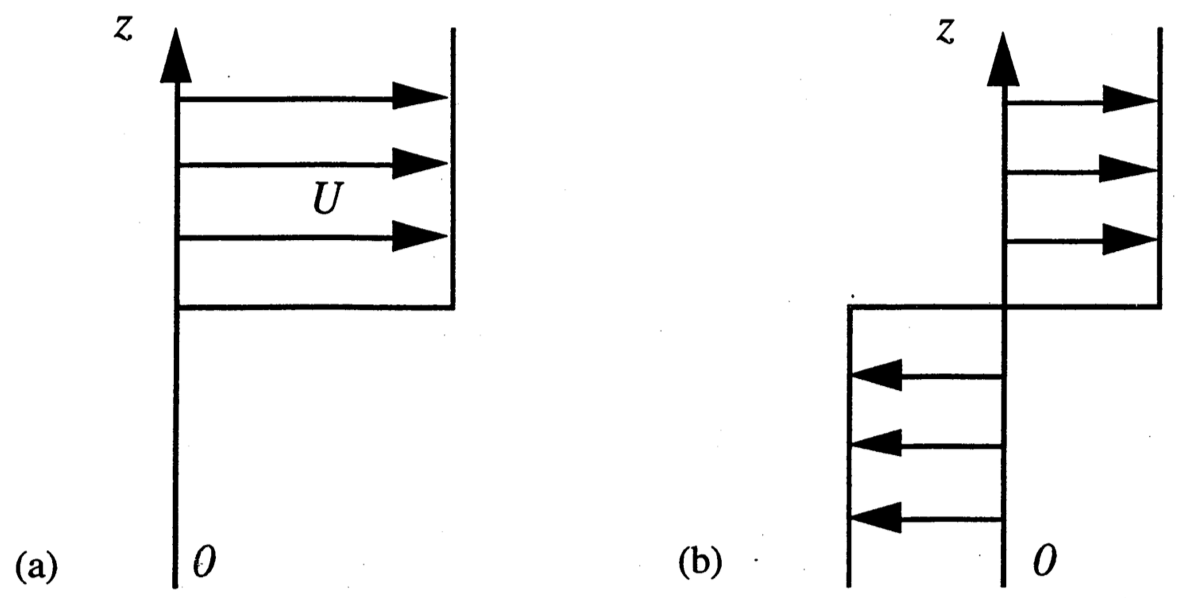


Fluid-velocity gradient normal to flow direction

Encountered in a wide range of flows of engineering interest:

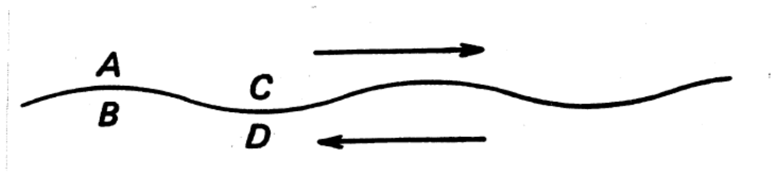
- Jets,
- Wakes,
- Boundary layers,
- Mixing layers,
- …

4 Shear-flow instability

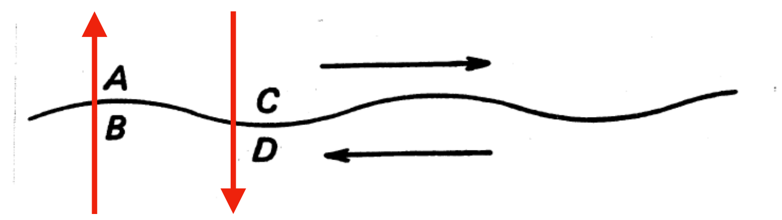


Basic mechanism can be understood in a simplified configuration

- **All vorticity concentrated on a line - vortex sheet**
- **Introduce a small stationary, wavy disturbance:**



4 Shear-flow instability



Flow accelerates at the crests, A & D, where streamlines converge,

Flow decelerates at the troughs, B & C, where streamlines diverge,

Tangential flow speed:

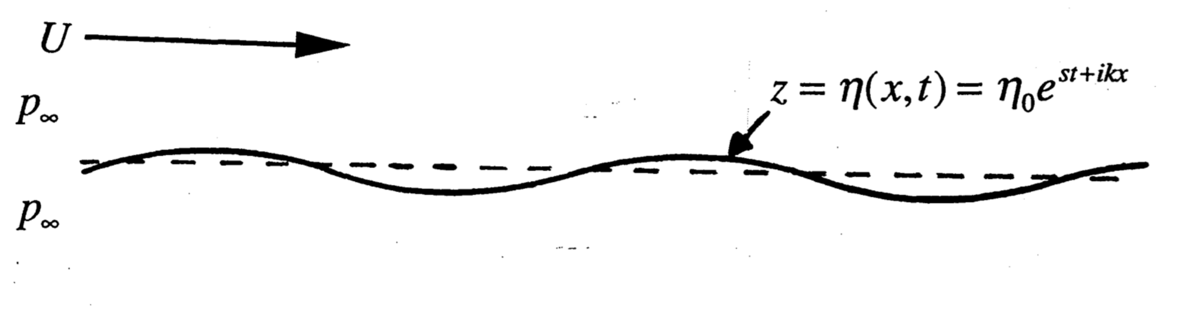
- greater at A than at B,
- greater at D than at C,

Bernoulli says:

- pressure at A less than at B
- force exerted from B to A and from C to D
- wave amplitude increases
- pressure difference increases
- normal force increases
- wave amplitude increases

INSTABILITY

Analysis using equations of motion



$$z = \eta(x,t) = \eta_0 \mathrm{e}^{st+ikx}$$

Above the vortex sheet we have potential flow; disturbance *Ansatz*:

$$\phi(x,z,t) = Ux + f(z)\mathrm{e}^{st+i\kappa x} \qquad z > \eta(x,t)$$

Potential flow -> Laplace's equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Differentiate disturbance Ansatz, $\phi(x, z, t) = Ux + f(z)\mathrm{e}^{st+i\kappa x}$**,**
twice with respect to x, y & z

$$-\kappa^2 f(z) + \frac{d^2 f(z)}{dz^2} = 0$$

A second-order ODE constraining the transverse structure of the flow.

$$-\kappa^2 f(z) + \frac{d^2 f(z)}{dz^2} = 0$$

Second-order ODE has general solution:

$$f(z) = A_1 \mathrm{e}^{-\kappa z} + A_2 \mathrm{e}^{\kappa z} \qquad z > \eta(x,t)$$
$$f(z) = B_1 \mathrm{e}^{-\kappa z} + B_2 \mathrm{e}^{\kappa z} \qquad z < \eta(x,t)$$

Transverse boundary conditions: velocity finite as $z \to \pm\infty$

$$A_2 = 0$$
$$B_1 = 0$$

$$\begin{aligned}\phi(x,z,t) &= Ux + f(z)\mathrm{e}^{st+i\kappa x} \\ &= Ux + A_1 \mathrm{e}^{st+i\kappa x-\kappa z} && z > \eta(x,t) \\ &= B_2 \mathrm{e}^{st+i\kappa x+\kappa z} && z < \eta(x,t)\end{aligned}$$

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Interface boundary conditions will determine A_1 **,** B_2 **&** $s(\kappa)$

Two kinds of interface boundary condition:

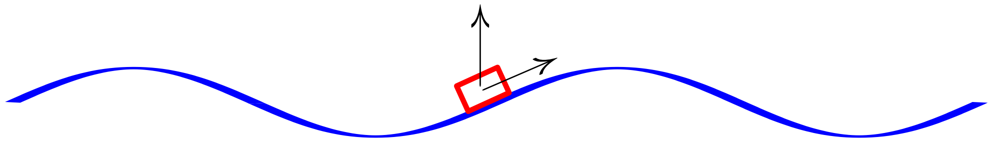
- **Kinematic BC: imposed by interface motion (moves with fluid),**
- **Dynamic BC: imposed by interface dynamics (momentum/pressure balance)**

Kinematic constraint:

- Consider transverse velocity of fluid particles at interface

Displacement: $\eta(x,t) = \eta_0 \mathrm{e}^{st+i\kappa x}$

Potential: $\phi(x,z,t) = Ux + A_1 \mathrm{e}^{st+i\kappa x-\kappa z}$



Normal velocity of fluid particles must match that of interface

$$\frac{\partial \phi}{\partial z} = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\eta$$

$$-A_1\kappa \mathrm{e}^{-\kappa\eta}\mathrm{e}^{st+i\kappa x} = (s+i\kappa U)\eta_0 \mathrm{e}^{st+i\kappa x}$$

$$-A_1\kappa \mathrm{e}^{-\kappa\eta} = (s+i\kappa U)\eta_0$$

Kinematic constraint:

$$-A_1\kappa \mathrm{e}^{-\kappa\eta} = (s + i\kappa U)\eta_0$$

Taylor series expansion of transverse structure

$$\mathrm{e}^{-\kappa\eta} = 1 - \kappa\eta + \ldots$$

Considering small disturbances the kinematic constraint reduces to

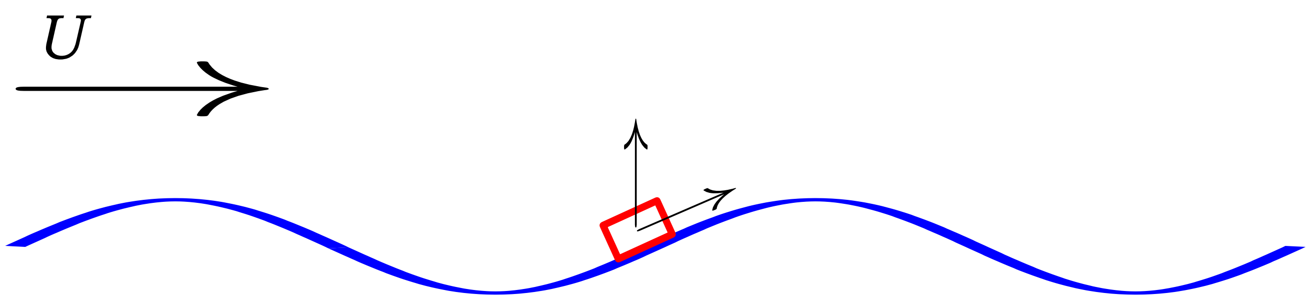
$$\begin{aligned} -A_1\kappa &= (s + i\kappa U)\eta_0 \\ B_2\kappa &= s\eta_0 \end{aligned}$$

Dynamic constraint

- Pressure is continuous across the interface
- Bernoulli's equation holds on either side, but not at the interface

At $\eta^+(x,t)$

$$\rho\frac{\partial\phi(x,z,t)|_{z=\eta^+}}{\partial t}+\frac{\rho}{2}\left(\frac{\partial\phi(x,z,t)|_{z=\eta^+}}{\partial x}\right)^2+p(x,\eta^+,t)+\rho g\eta^+(x,t)=\text{constant}$$
$$=p_\infty+\frac{\rho U^2}{2}$$



Unsteady Bernoulli equation

$$\rho \frac{\partial \phi(x,z,t)|_{z=\eta^+}}{\partial t} + \frac{\rho}{2}\left(\frac{\partial \phi(x,z,t)|_{z=\eta^+}}{\partial x}\right)^2 + p(x,\eta^+,t) + \rho g \eta^+(x,t) = p_\infty + \frac{\rho U^2}{2}$$

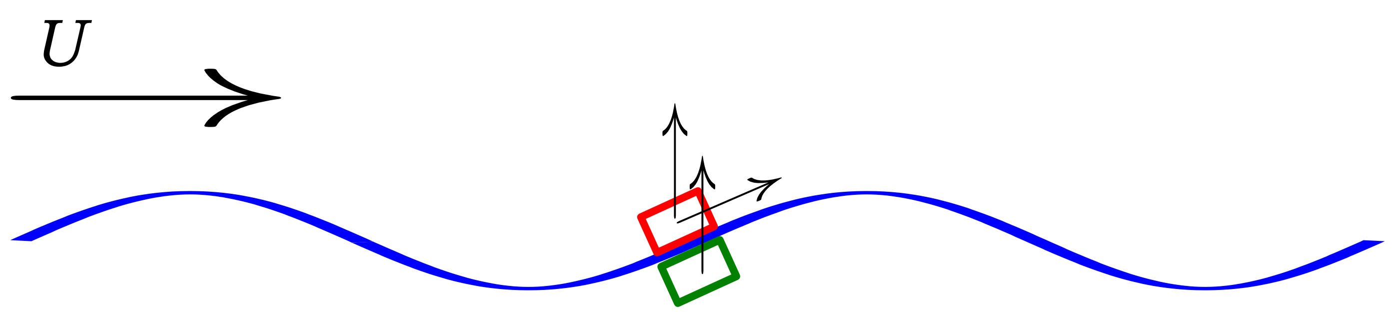
Perturbation *Ansatz*: $\phi(x,z,t) = Ux + A_1 \mathrm{e}^{st+i\kappa x-\kappa z}$

$$\left(\frac{\partial \phi(x,z,t)}{\partial x}\Big|_{z=\eta^+}\right)^2 = U^2 + 2Ui\kappa A_1 \mathrm{e}^{-\kappa\eta^+}\mathrm{e}^{st+i\kappa x} + \text{non-linear terms}$$

$$\frac{\partial \phi(x,z,t)}{\partial t}\Big|_{z=\eta^+} = sA_1 \mathrm{e}^{-\kappa\eta^+}\mathrm{e}^{st+i\kappa x}$$

$$\begin{aligned} p(x,\eta^+,t) &= p_\infty - \rho(s+i\kappa U)A_1 \mathrm{e}^{-\kappa\eta^+}\mathrm{e}^{st+i\kappa x} - \rho g \eta_0 \mathrm{e}^{st+i\kappa x} \\ &= p_\infty - \rho(s+i\kappa U)A_1 \mathrm{e}^{st+i\kappa x} - \rho g \eta_0 \mathrm{e}^{st+i\kappa x} \end{aligned}$$

$$p(x, \eta^+, t) = p_\infty - \rho(s + i\kappa U) A_1 \mathrm{e}^{st+i\kappa x} - \rho g \eta_0 \mathrm{e}^{st+i\kappa x}$$

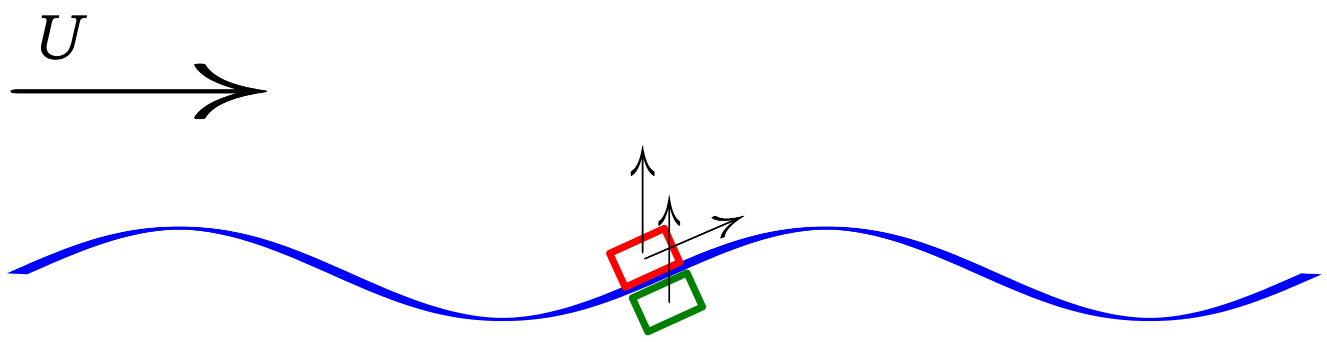


$\eta^-(x, t)$

$$p(x, \eta^-, t) = p_\infty - \rho s B_2 \mathrm{e}^{st+ikx} - \rho g \eta_o \mathrm{e}^{st+ikx}$$

Matching across vortex sheet

$$sB_2 = (s + i\kappa U) A_1$$



Dynamic constraint

$$sB_2 = (s + i\kappa U)A_1$$

Recall kinematic constraint

$$-A_1\kappa = (s + i\kappa U)\eta_0$$

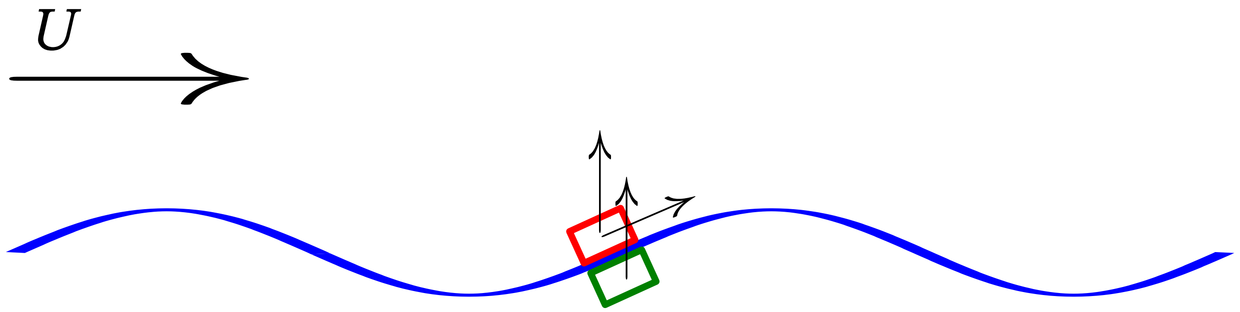
$$B_2\kappa = s\eta_0$$

Combining constraints

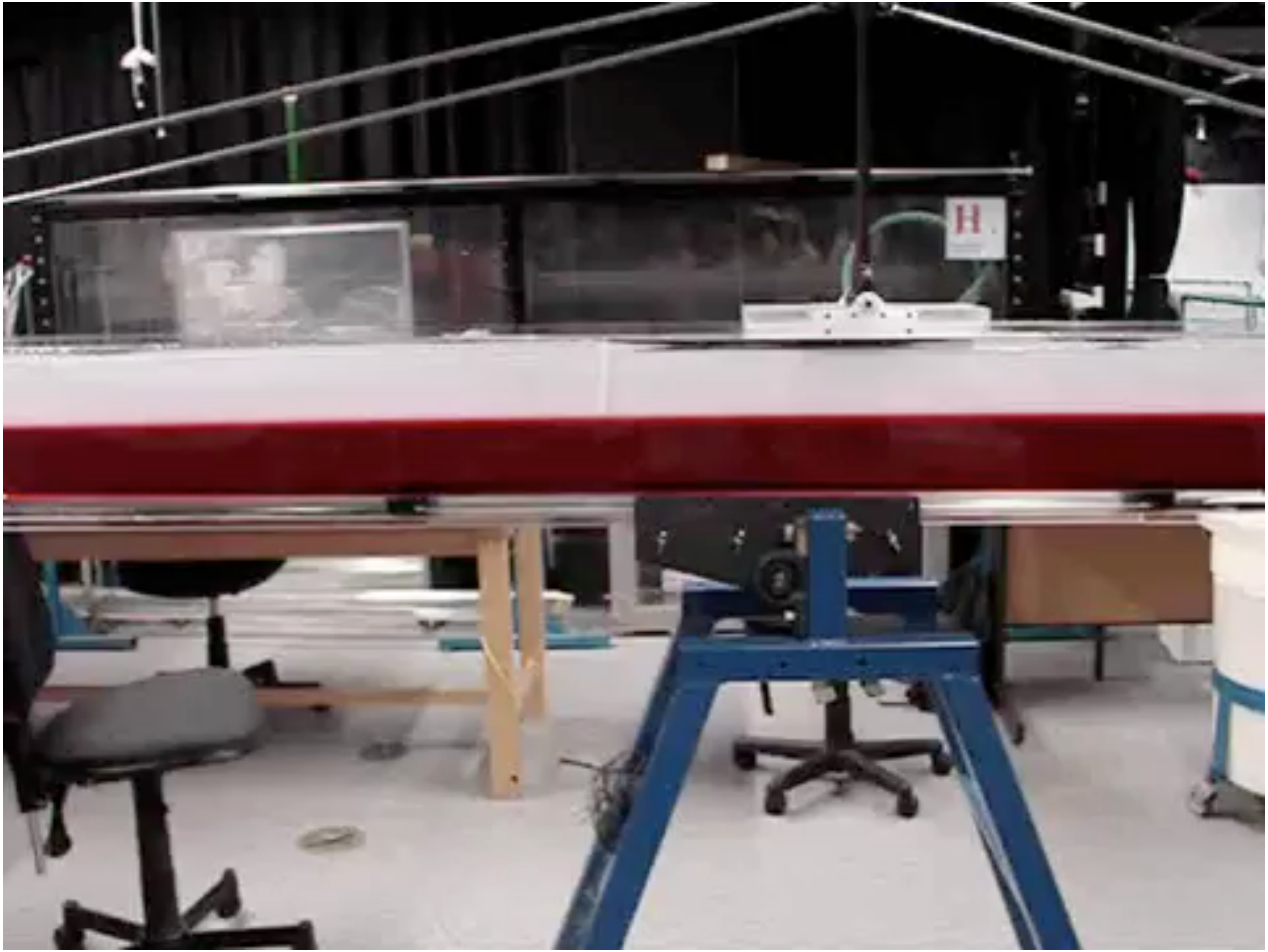
$$s^2 + (s + i\kappa U)^2 = 0$$

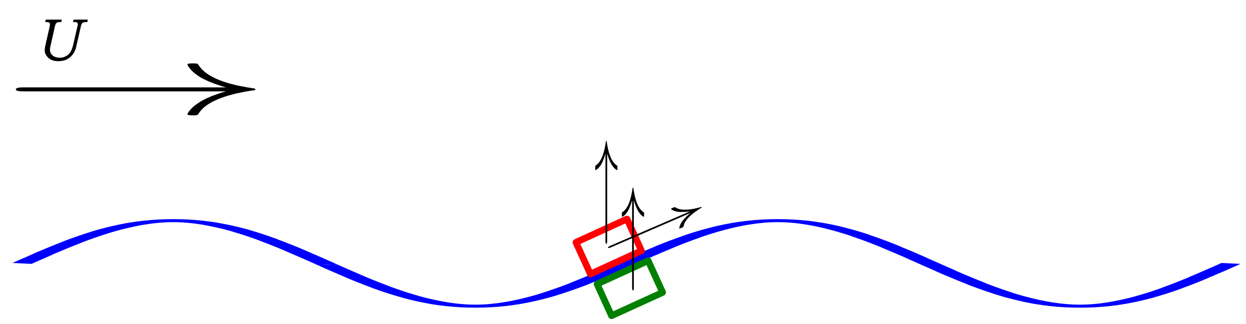
$$s^2 + (s + i\kappa U)^2 = 0$$

$$s = -\frac{1}{2} i\kappa U \pm \frac{1}{2}\kappa U$$



$$\eta(x,t) = \eta_0 \mathrm{e}^{\frac{1}{2}\kappa U t + i\kappa(x - \frac{1}{2}Ut)}$$





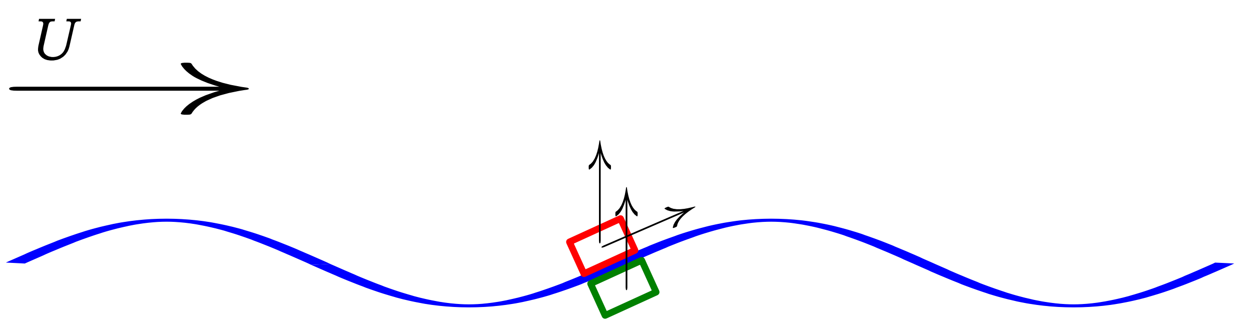
$$\eta(x,t) = \eta_0 \mathrm{e}^{\frac{1}{2}\kappa U t + i\kappa(x - \frac{1}{2}Ut)}$$

For each wavenumber, *k*, there is an unstable mode travelling in the *x*-direction With phase speed *U/2* and growing exponentially with growth rate *kU/2*

In the case of finite thickness only a certain range of wavenumber will be unstable

The spatial stability problem can be considered by letting $s=i\omega$ **in**

$$s^2 + (s + i\kappa U)^2 = 0$$



$$s^2 + (s + i\kappa U)^2 = 0$$

Spatial problem

$$U^2\kappa + 2\omega U\kappa + \omega^2 = 0$$

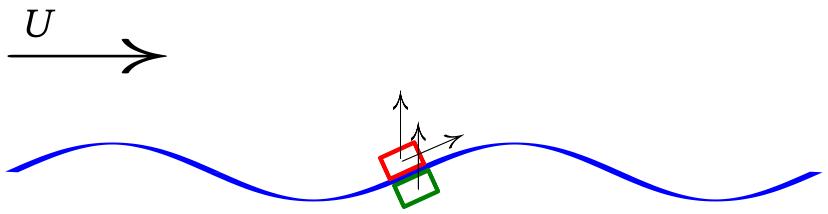
Roots

$$\kappa = -\frac{\omega}{U} \pm i\frac{\omega}{U}$$

Write down space-time behaviour of the spatial instability.

4 Shear-flow instability

Résumé



Potential flow assumed above and below the vortex sheet: Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Disturbance *Ansatz*: velocity potential with normal modes:

$$\phi(x, z, t) = Ux + f(z)\mathrm{e}^{st + i\kappa x}$$

Leads to ODE for transverse structure:

$$-\kappa^2 f(z) + \frac{d^2 f(z)}{dz^2} = 0$$

General solution:

$$f(z) = A_1 \mathrm{e}^{-\kappa z} + A_2 \mathrm{e}^{\kappa z}$$
$$f(z) = B_1 \mathrm{e}^{-\kappa z} + B_2 \mathrm{e}^{\kappa z}$$

Boundary and interface matching conditions:

$$\eta(x, t) = \eta_0 \mathrm{e}^{\frac{1}{2}\kappa U t + i\kappa (x - \frac{1}{2} U t)}$$

Summary

Can assess stability of system by considering energy before and after introduction of a perturbation

Alternatively one can consider linear dynamics of problem

For vortex-sheet problem:

- Simplification of governing equations: Laplace & Bernoulli
- Introduction of normal modes in ***x*** and *t* $\phi(x,z,t) = Ux + f(z)\mathrm{e}^{st+i\kappa x}$
- Solution for *s(k)*
- Transverse structure can also be determined (eigenfunction) which means that the entire space-time structure is obtained

Analytical solution possible for vortex sheet: exception rather than rule: numerical solution is usually necessary, but rationale is the same: reduce PDE system to ODE system that takes form of an eigenvalue problem