Master Turbulence 2022-2023

Advanced Signal Processing – Part I

(ASP1)

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Download the course: ***http://www.pprime.fr/marx-david***

Course - General outlook and Grade

LECTURES [38h]

- Part I [14h, David Marx]
- Part II [24h, Jean-Christophe Valière]

LABS [12h]

- 3 Labs in matlab, 4h each, all related to part I [David Marx]
  → 3 reports in pdf format to be returned by email within the three weeks following the lab
  (respect instructions for file naming !)

GRADE

- One mark for the labs (will be the mark for Part I of the course)
- One mark for the exam (will be the mark for Part II of the course)
- Final grade : 1/3 Exam Mark + 2/3 Lab Mark

PART I- General outlook

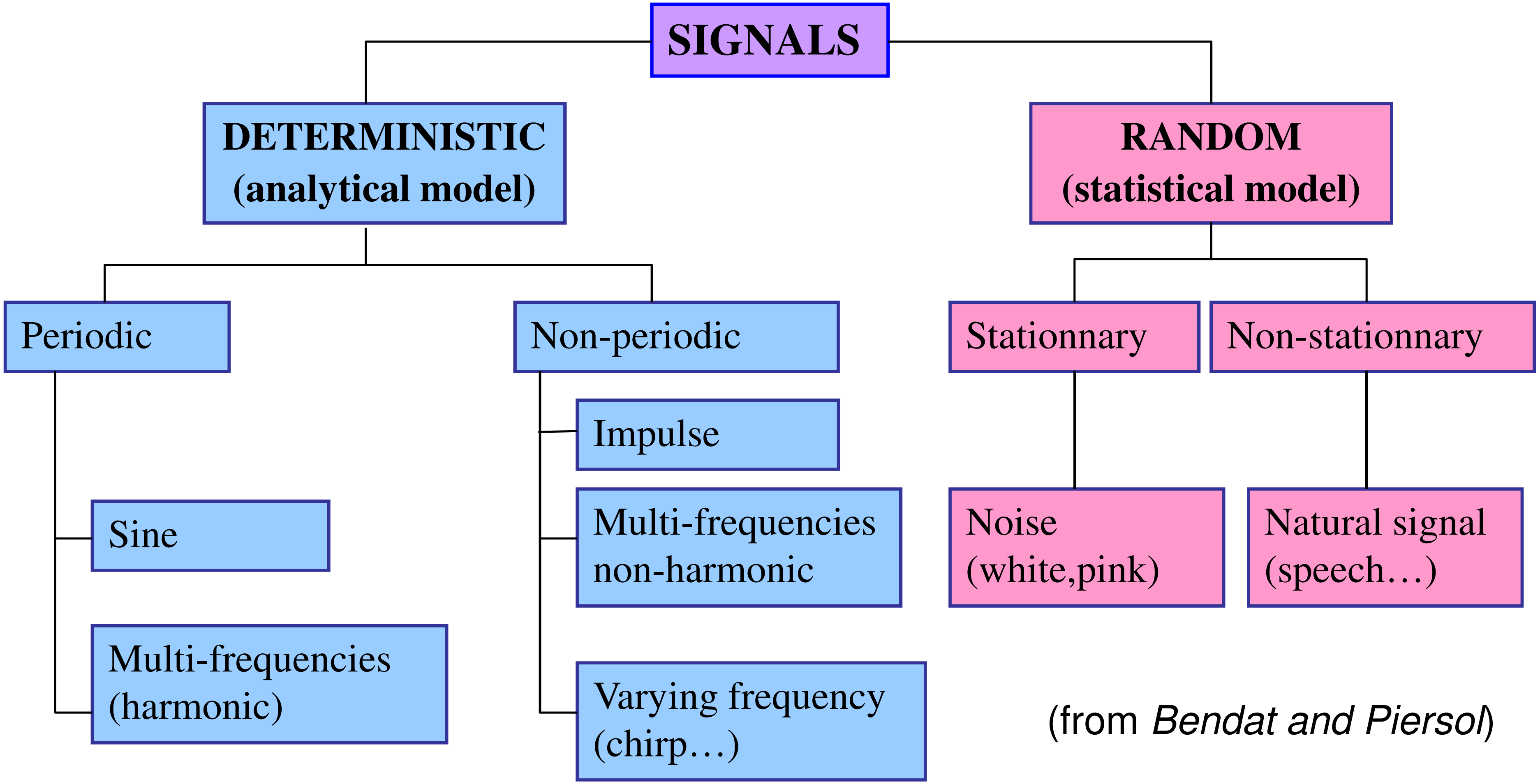
7 lectures [7 * 2h] + 3 labs with matlab [3 * 4h]:

- **Lecture 1 : Fourier Transform**
- **Lecture 2 : Discrete Fourier Transform**
- **Lecture 3 : Introduction to Random Processes**
- **Lab 1: Welch Periodogram and application to hot wire measurements**

- **Lecture 4 : Time-frequency analysis 1 - Introduction**
- **Lecture 5 : Time-frequency analysis 2 - Distributions**
- **Lecture 6 : Time-frequency analysis 3 – Wavelets**
- **Lab 2: Continuous Wavelet Transform (CWT)**

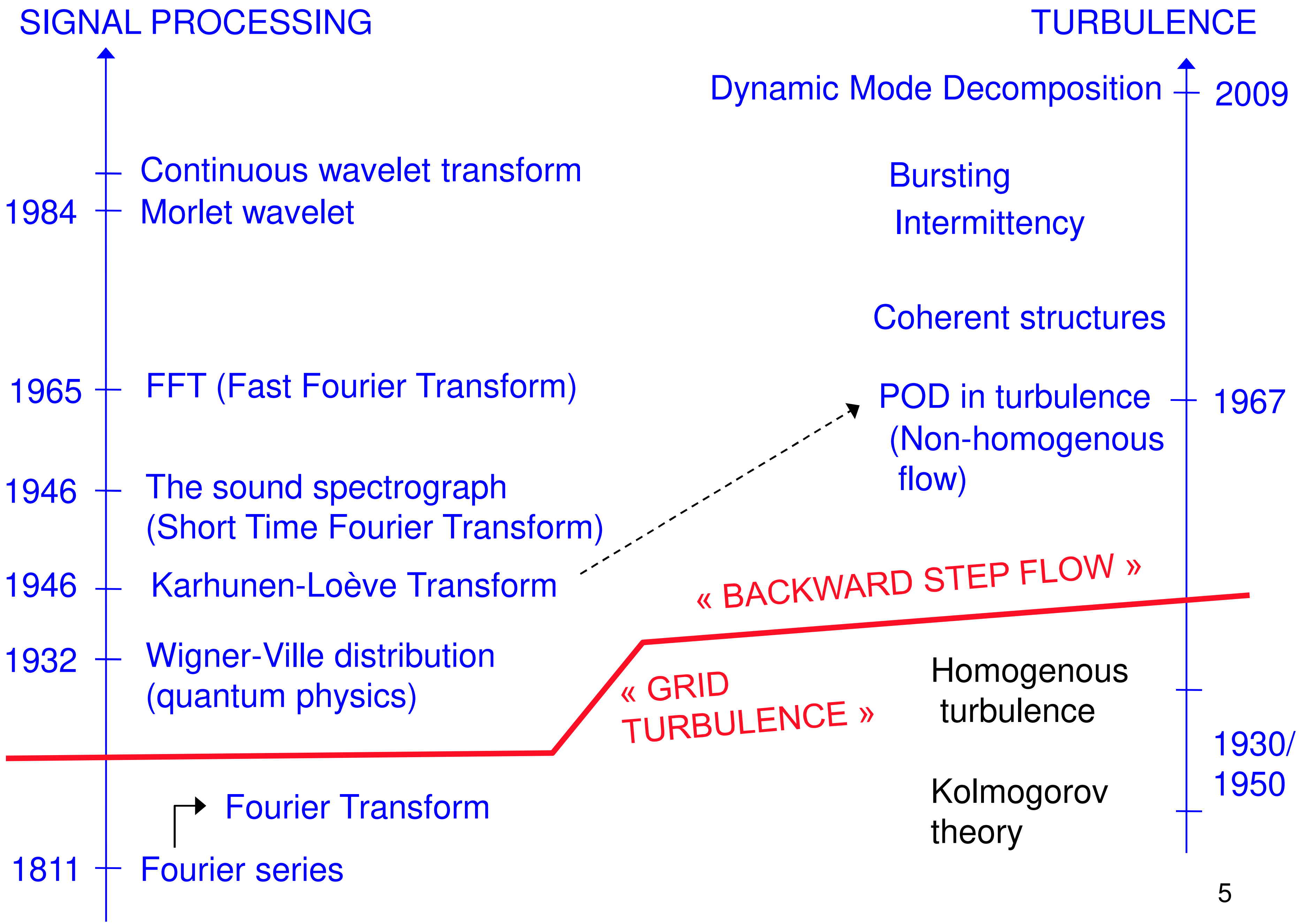
- **Lecture 7 : Proper Orthogonal Decomposition (POD)**
- **Lab 3: POD analysis of a flow (shear-layer)**

Introduction - Signal classification



(from *Bendat and Piersol*)

- Usually signals belong to several boxes
- Another property is whether the signal has *finite energy* or *finite power*


SIGNAL PROCESSING
TURBULENCE
Dynamic Mode Decomposition
2009
Continuous wavelet transform
1984
Morlet wavelet
Bursting
Intermittency
Coherent structures
1965
FFT (Fast Fourier Transform)
POD in turbulence
1967
(Non-homogenous
flow)
1946
The sound spectrograph
(Short Time Fourier Transform)
1946
Karhunen-Loève Transform
« BACKWARD STEP FLOW »
1932
Wigner-Ville distribution
(quantum physics)
« GRID
TURBULENCE »
Homogenous
turbulence
1930/
1950
Fourier Transform
Kolmogorov
theory
1811
Fourier series

DNM Turbulence

Advanced Signal Processing

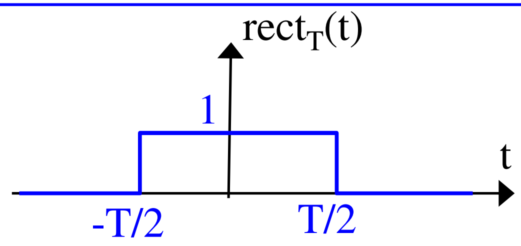
Lectures 1 & 2 : Fourier Transform

Part 1
The Continuous (analog) signals
The Fourier Transform

Dirac delta function

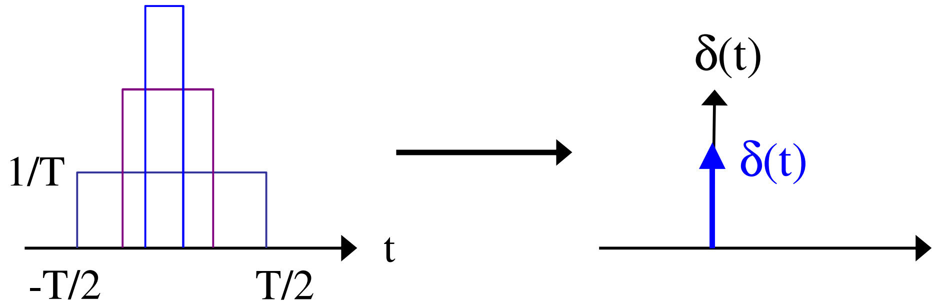
Rectangular function

$$rect_T(t) = \begin{cases} 1 & \text{if } -T/2 \le t \le T/2 \\ 0 & otherwise \end{cases}$$



Dirac delta function

$$\delta(t) = \lim_{T \to 0}\left(\frac{1}{T} rect_T(t)\right)$$



Properties:

1. $\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$ or $\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$

2. $f(t) \cdot \delta(t - t_0) = f(t_0) \cdot \delta(t - t_0) \neq f(t_0)$

3. $f(t) * \delta(t - t_0) = f(t - t_0)$ *See convolution later.*

Remark: actually, the Dirac delta function is not a function, it is a generalized function / a distribution. To get rid of δ, integration is needed!

I. Fourier Transform

The signal as a sum of waves

In harmonic (Fourier) analysis, a signal is a sum of complex waves $e^{j2\pi ft}$ at frequency f.

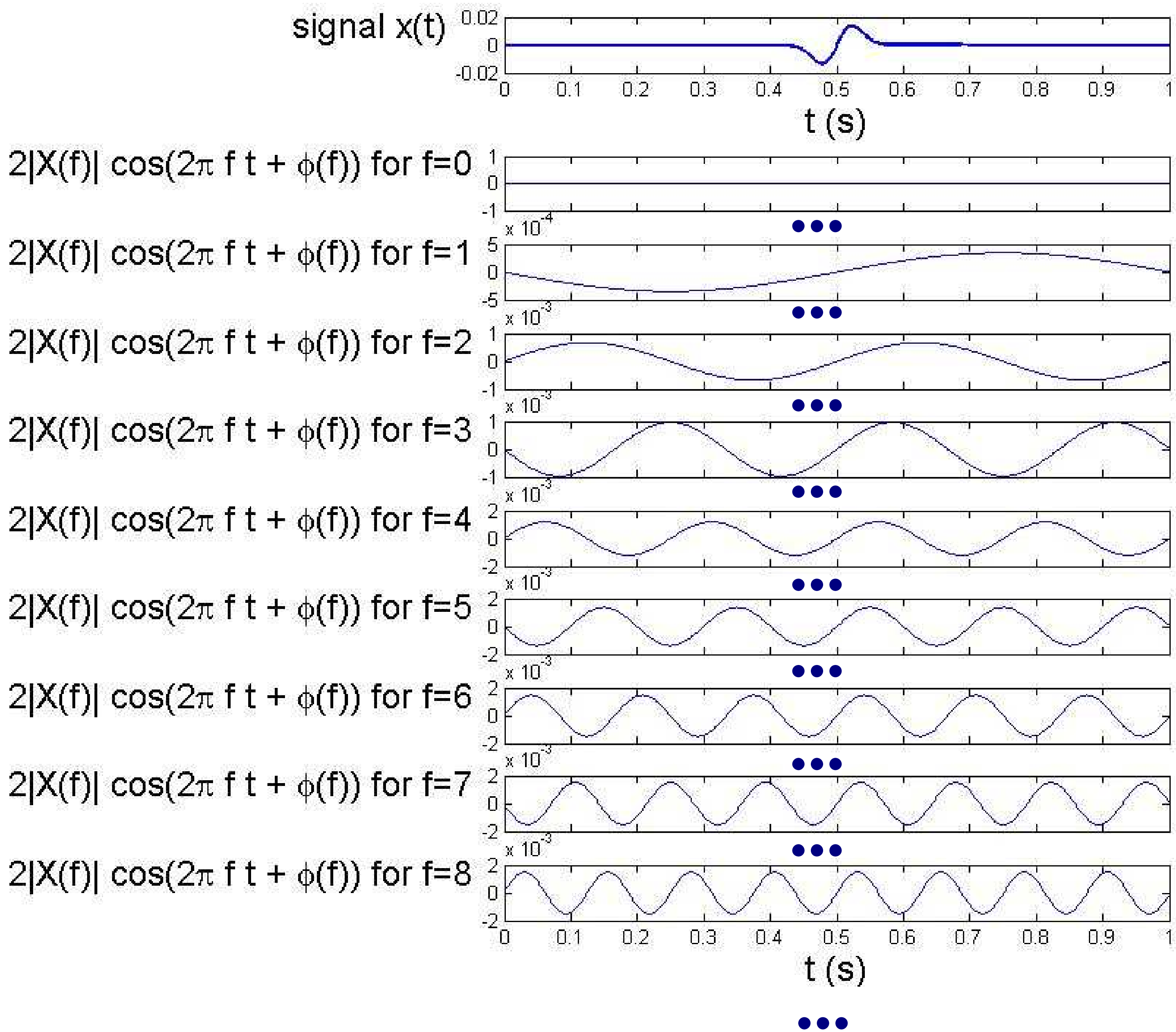
This is expressed by the Inverse Fourier Transform (IFT):

$$x(t) = \text{IFT}[X] = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

Amplitude, Phase

$$X(f) = |X(f)|e^{j\varphi(f)}$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df = \int_{-\infty}^{\infty} |X(f)|e^{j(2\pi ft+\varphi(f))}df$$


signal x(t)
0.02
0
-0.02
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
t (s)
2|X(f)| cos(2π f t + φ(f)) for f=0
2|X(f)| cos(2π f t + φ(f)) for f=1
x 10^-4
2|X(f)| cos(2π f t + φ(f)) for f=2
x 10^-3
2|X(f)| cos(2π f t + φ(f)) for f=3
x 10^-3
2|X(f)| cos(2π f t + φ(f)) for f=4
x 10^-3
2|X(f)| cos(2π f t + φ(f)) for f=5
x 10^-3
2|X(f)| cos(2π f t + φ(f)) for f=6
x 10^-3
2|X(f)| cos(2π f t + φ(f)) for f=7
x 10^-3
2|X(f)| cos(2π f t + φ(f)) for f=8
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
t (s)

How to obain the complex amplitude *X(f)*?

The Fourier Transform (FT) is defined by:

sign reversal compared to IFT

$$X(f) = \mathrm{FT}[x] = \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt$$

Note: to know how much the wave at frequency *f* is contained in *x*, *x* is multiplied by the complex conjugated wave, and the product is integrated.

Case of a real signal:

For a real signal, the amplitudes of the waves at *f* and –*f* are related:

$$X^*(f) = X(-f)$$

$$\left|X^*(f)\right| = \left|X(f)\right| = \left|X(-f)\right|$$

$$\phi(-f) = -\phi(f)$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi f t}df = \int_{0}^{\infty} 2\left|X(f)\right|\cos(2\pi f t + \phi(f))df$$

For which signals is possible to compute the FT?

The Fourier Transform AND its inverse are defined for signals *x(t)* in L^2 (square integrable signals), that is for signals with a finite energy (signals observed experimentally are always windowed somehow and fall into this category).

Energy of the signal

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Energy conservation

- Parseval's equality:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

« The energy of a signal is the sum of the energies contained in its waves. »

FT of some signals not in L^2 using distributions

- The Fourier Transform and its inverse may also be defined for some signals that are not in L^2. This is often possible using distributions (such as the Dirac delta function).

- For example:

$$x(t) = e^{j2\pi f_0 t}$$

is not in L^2 (not in L^1 either). Its Fourier Transform exists and is:

$$X(f) = FT[x(t)] = \delta(f - f_0)$$

Note: this is **not**:
$X(f_0)=1$;
$X(f \neq f_0)=0$

- For the corresponding real signal:

$$x(t) = \cos(2\pi f_0 t)$$

the Fourier Transform is:

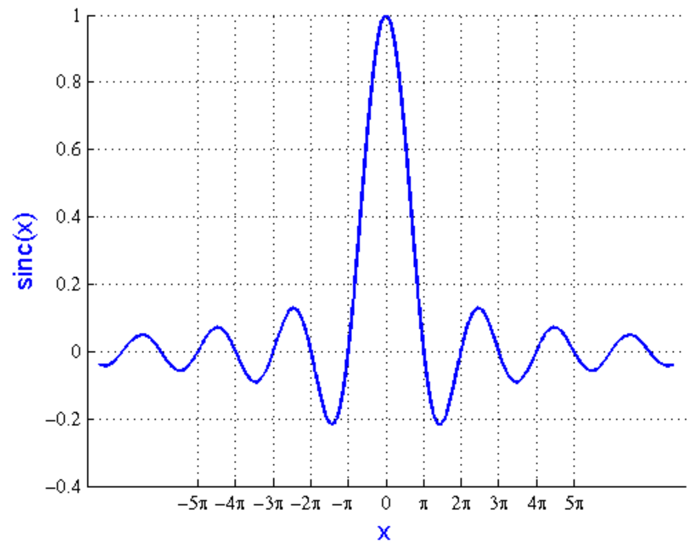
$$X(f) = FT[x(t)] = \frac{1}{2}[\delta(f + f_0) + \delta(f - f_0)]$$

Classical FTs

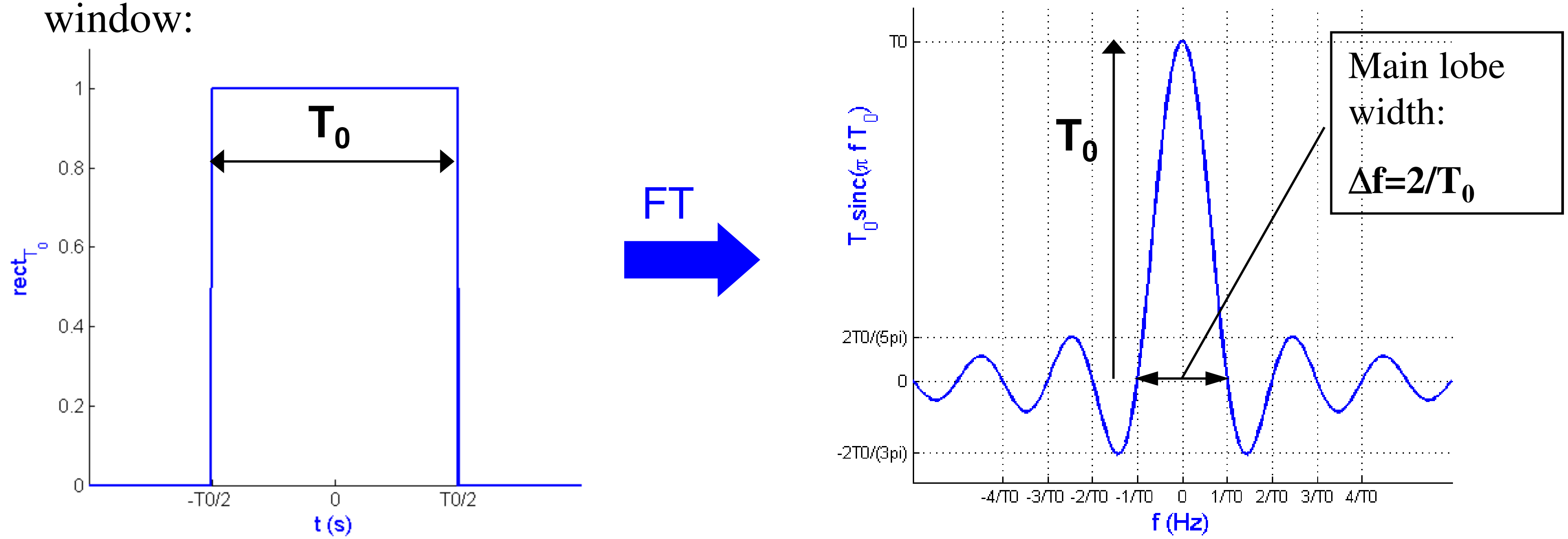
	Signal	FT
Constant	1	$\delta(f)$
Dirac Delta function	$\delta(t)$	1
Time translated impulse	$\delta(t - t_0)$	$e^{-j2\pi f t_0}$
Complex exponential	$e^{j2\pi f_0 t}$	$\delta(f - f_0)$
Cosine	$\cos(2\pi f_0 t)$	$\frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$
Sine	$\sin(2\pi f_0 t)$	$\frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$
Dirac Comb	$\sum_{k=-\infty}^{+\infty} \delta(t - kT)$	$\frac{1}{T} \sum_{k=-\infty}^{+\infty} \delta\left(f - \frac{k}{T}\right)$
Rectangular window	$rect_{T_0}(t)$	$T_0 \frac{\sin(\pi f T_0)}{\pi f T_0} = T_0 \text{sinc}(\pi f T_0)$
Gaussian	$e^{-\alpha^2 t^2}$	$\frac{\sqrt{\pi}}{\alpha} e^{-\pi^2 \frac{f^2}{\alpha^2}}$

The sinc function

$$sinc(t) = \begin{cases} 1 \ if \ t = 0 \\ \dfrac{\sin(t)}{t} \ otherwise \end{cases}$$



• The sinc is very important in practice because it is the FT of the rectangular window:

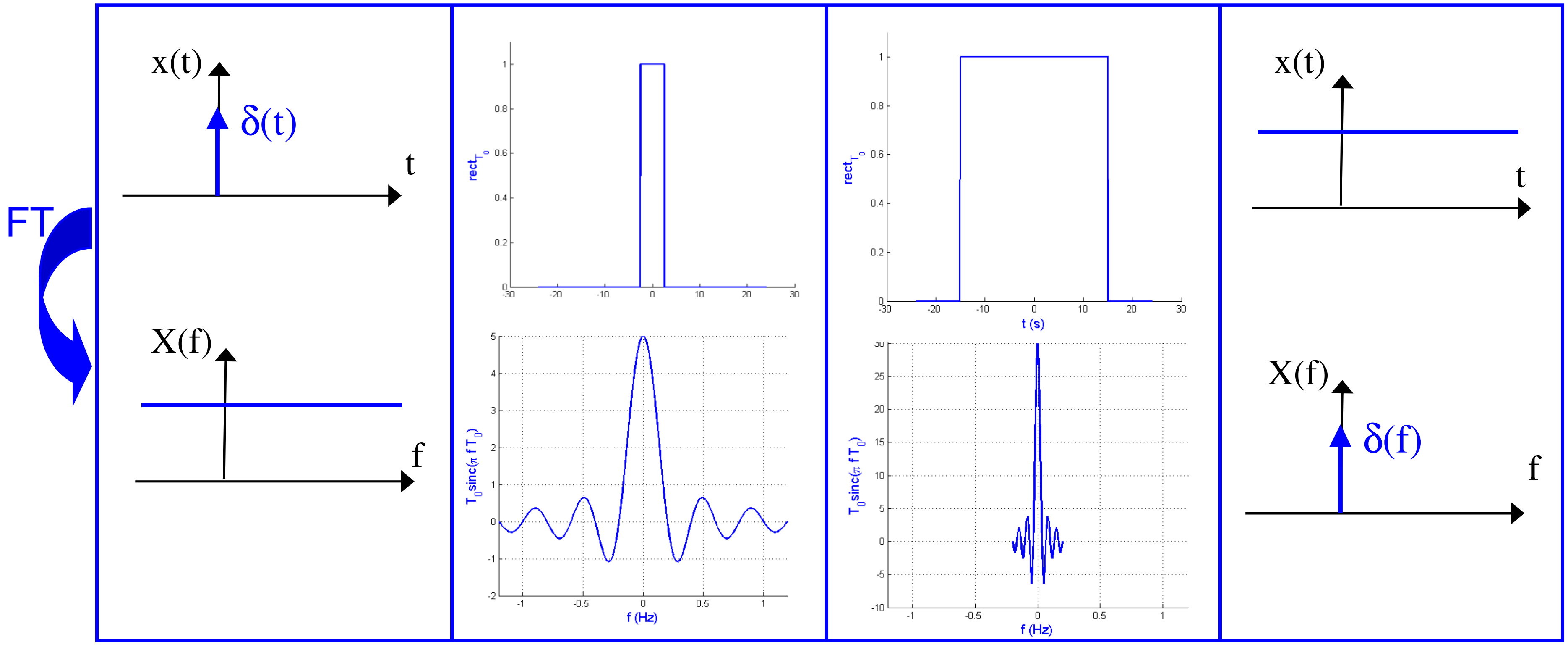


• The more the rectangular window is wide (T_0 large), the more its FT has a fine lobe ($\Delta f = 2/T_0$ is small) and a large amplitude (T_0). When T_0 goes to infinity the sinc tends towards a Dirac.

Some properties of the FT

	Signal	Its FT
Linearity	$g(t)+h(t)$	$G(f)+H(f)$
Translation in time	$h(t+\tau)$	$H(f)e^{j2\pi f\tau}$
Modulation (translation freq.)	$h(t)e^{j2\pi f_0 t}$	$H(f-f_0)$
Dilatation (k<1)-contraction (k>1)	$h(kt)$	$(1/\|k\|)H(f/k)$
Time reversal	$g(t)=h(-t)$	$G(f)=H^*(f)$
Complex conjugate	$g(t)=h^*(t)$	$G(f)=H^*(-f)$
Real signal	$h(t)$	$H(-f)=H^*(f)$
Parity	Signal: real even	TF: real even
	Signal: real odd	TF: imaginary odd
Time derivative	$g(t)=dh/dt$	$G(f)=j2\pi fH(f)$
n[th] time derivative	$g(t)=d^n h/dt^n$	$G(f)=(j2\pi f)^n H(f)$

Dilatation property



• The shortest the signal is (in time), the more widely spread it is in the spectral space. A more formal way to account for this is the Heisenberg-Gabor principle.

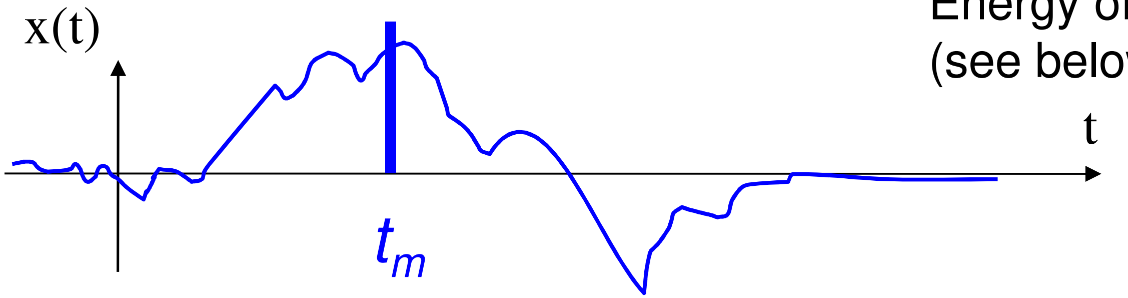
Heisenberg-Gabor Principle

- Localization of the signal in the time domain:

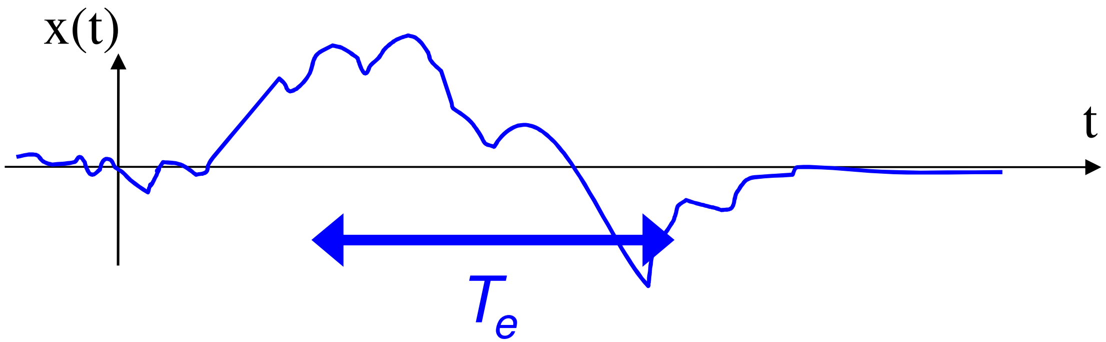
Average time: $$t_m = \frac{1}{E_x}\int_{-\infty}^{\infty} t|x(t)|^2 dt$$

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Energy of the signal (see below)



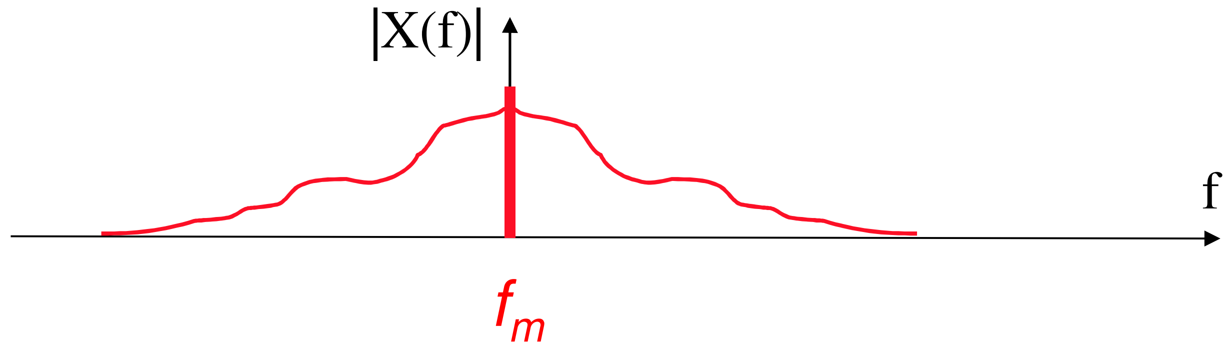
Effective duration: $$T_e{}^2 = \frac{1}{E_x}\int_{-\infty}^{\infty} (t - t_m)^2 |x(t)|^2 dt$$



- Localization of the signal in the frequency domain:

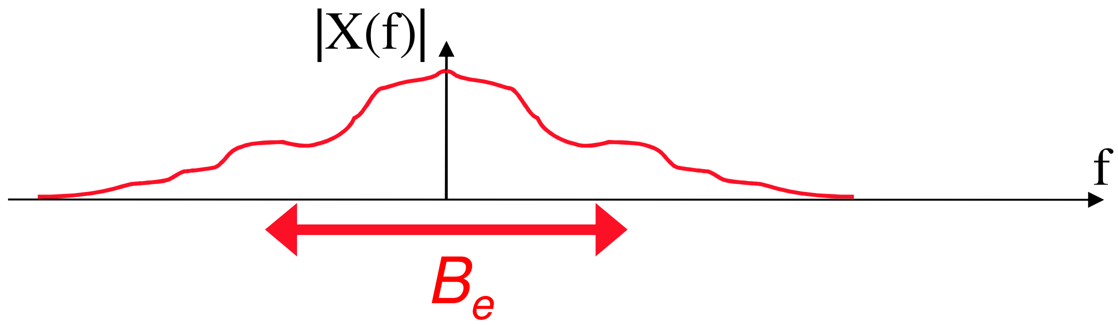
Average frequency:

$$f_m = \frac{1}{E_x}\int_{-\infty}^{\infty} f\,|X(f)|^2 df$$



Effective bandwith:

$$B_e^{\ 2} = \frac{1}{E_x}\int_{-\infty}^{\infty} (f - f_m)^2 |X(f)|^2 df$$



• The Heisenberg-Gabor principle states that the *duration-bandwith product* should satisfy:

$$T_e B_e \geq \frac{1}{4\pi}$$

This means that a signal that is well localized in time (small T_e) is not well localized in frequency (large B_e). And conversely.

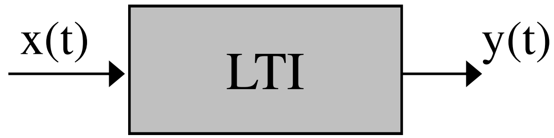
Exercise: show that the equality $T_e B_e = 1/(4\pi)$ is met for a gaussian signal.

Hint: Use the FT of a gaussian signal given in the table and the definition of the energy E_x given later. Use parity to show that $t_m=0$, $f_m=0$. Use integration by parts and the Gauss Integral:

$$\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$$

II. Linear Time Invariant systems
Convolution operator, Impulse Response

Linear Time Invariant system:



- The input and the output are linked by a linear differential equation whose coefficients are invariant in time:

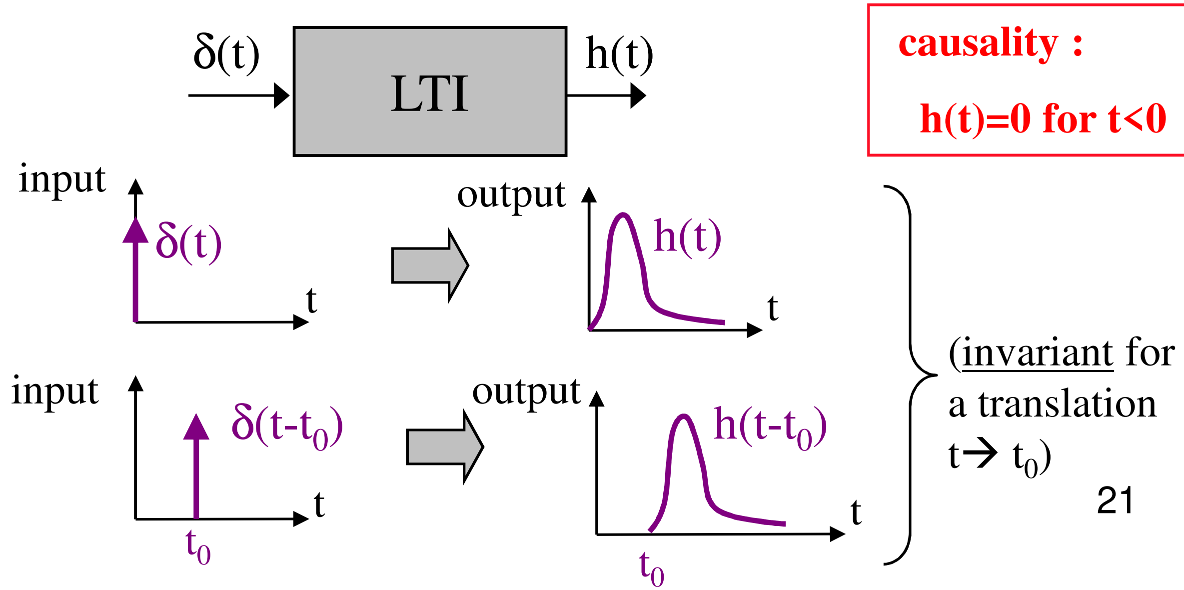
$$F\left(y(t), \frac{dy(t)}{dt}, ..., \frac{d^{N-1}y(t)}{dt^{N-1}}, x(t), \frac{dx(t)}{dt}, ..., \frac{d^{M-1}x(t)}{dt^{M-1}} \right) = 0$$

- A lot of systems, but not all of them!

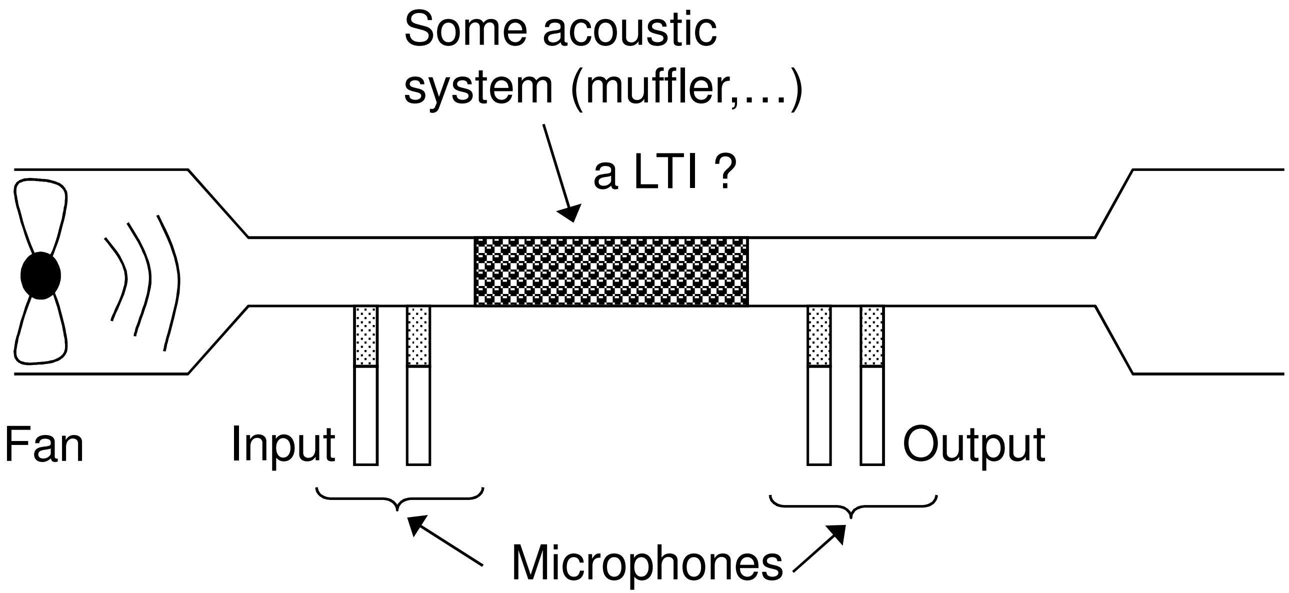
Impulse Response (IR):

causality :

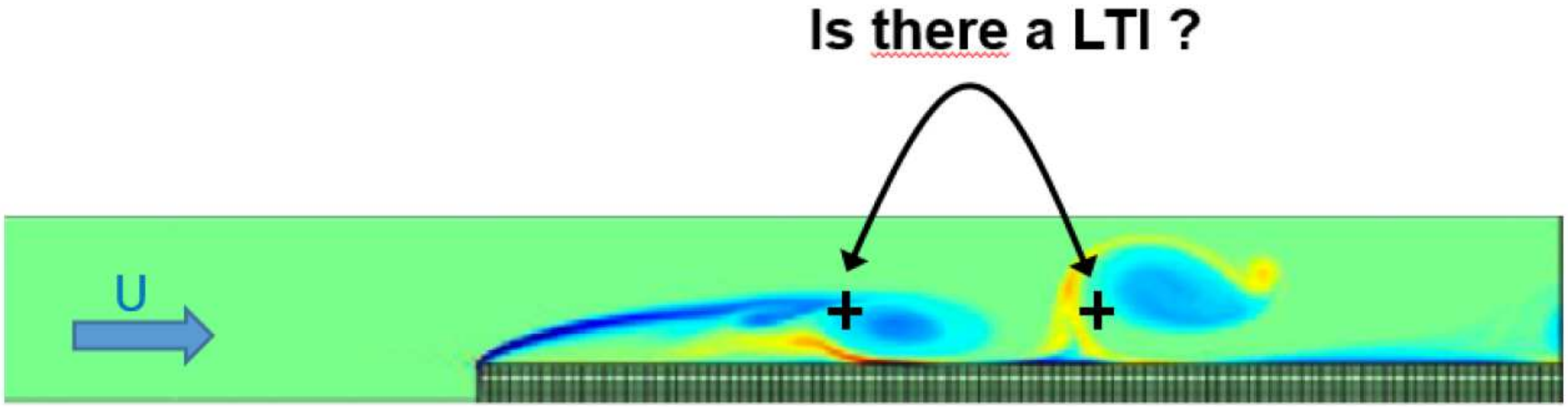
h(t)=0 for t<0



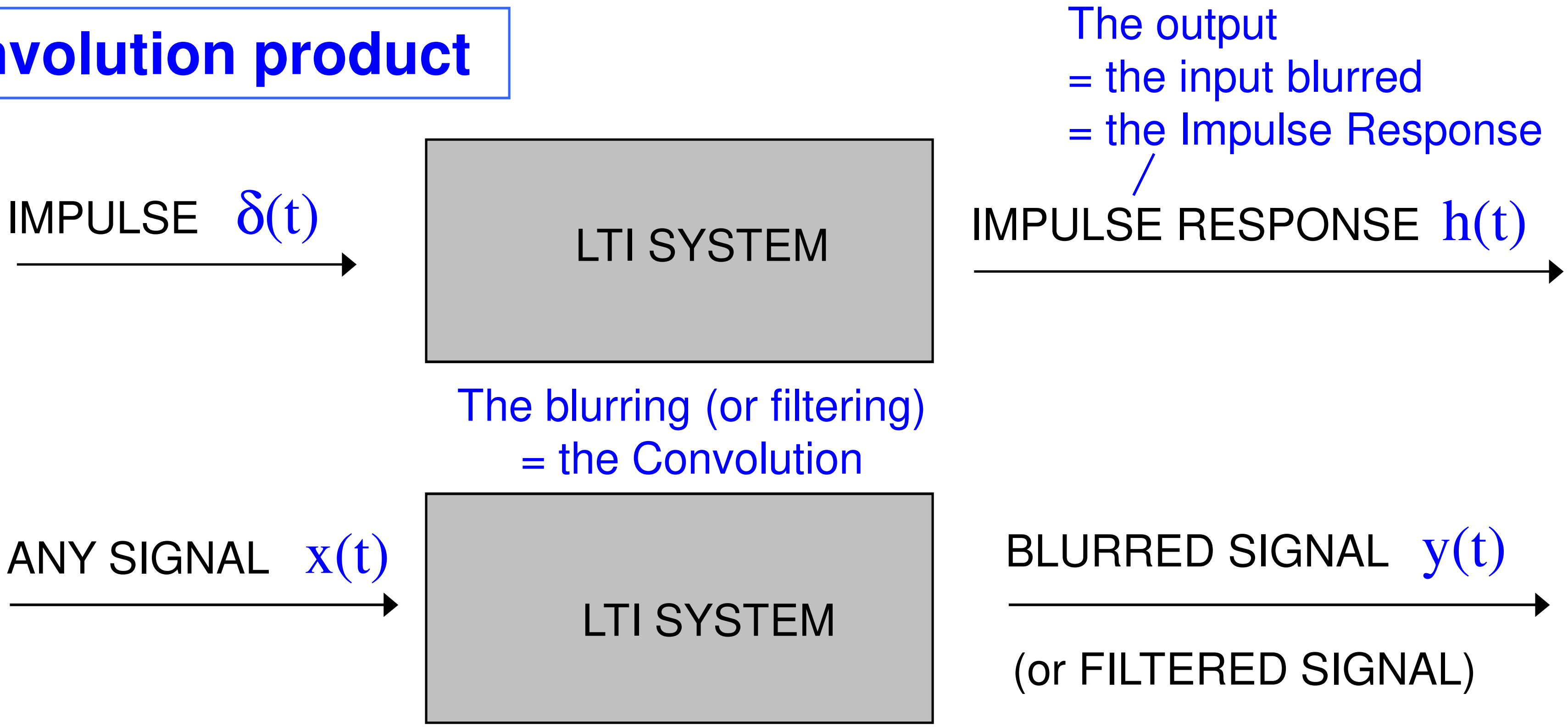
Example 1 - System characterization in duct acoustics



Example 2 – Convection of vortices in turbulence



Convolution product



Convolution operator and impulse response

The IR allows to calculate the response to an arbitrary input using the convolution product. For an input x(t), the output y(t) is given by:

Convolution product

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

The convolution product is commutative: $f * g = g * f$

Note on the calculation of the convolution product:

One wants to calculate the output y(t), given by:

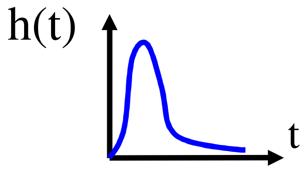
$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

Normally, *h* is causal : *h(τ) =0* for *τ<0*, the integral becomes:

$$y(t) = x(t) * h(t) = \int_{0}^{+\infty} h(\tau)x(t-\tau)d\tau$$

Note that for every single time *t*, an integral must be calculated for *y(t)*. In that integral, you need to take into account the inputs at all the times that precede *t*. This is so because all the inputs *x(t-τ)* are needed and *t-τ<t* for *τ≥0*. Moreover, the values *x(t-τ)* enter the integral with a weighting factor *h(τ)*. Hence, for the output at time *t*, some times of the input are more important than others: the larger *h(τ)*, the larger the importance of *x(t-τ)* in *y(t)*.

Very often the impulse response *h(τ)* becomes small when *τ→∞* or more simply for *τ* large enough. It then looks like this:



Then the output depends mainly on the input at instants «close to» *t* (and preceding *t* when *h* is causal).

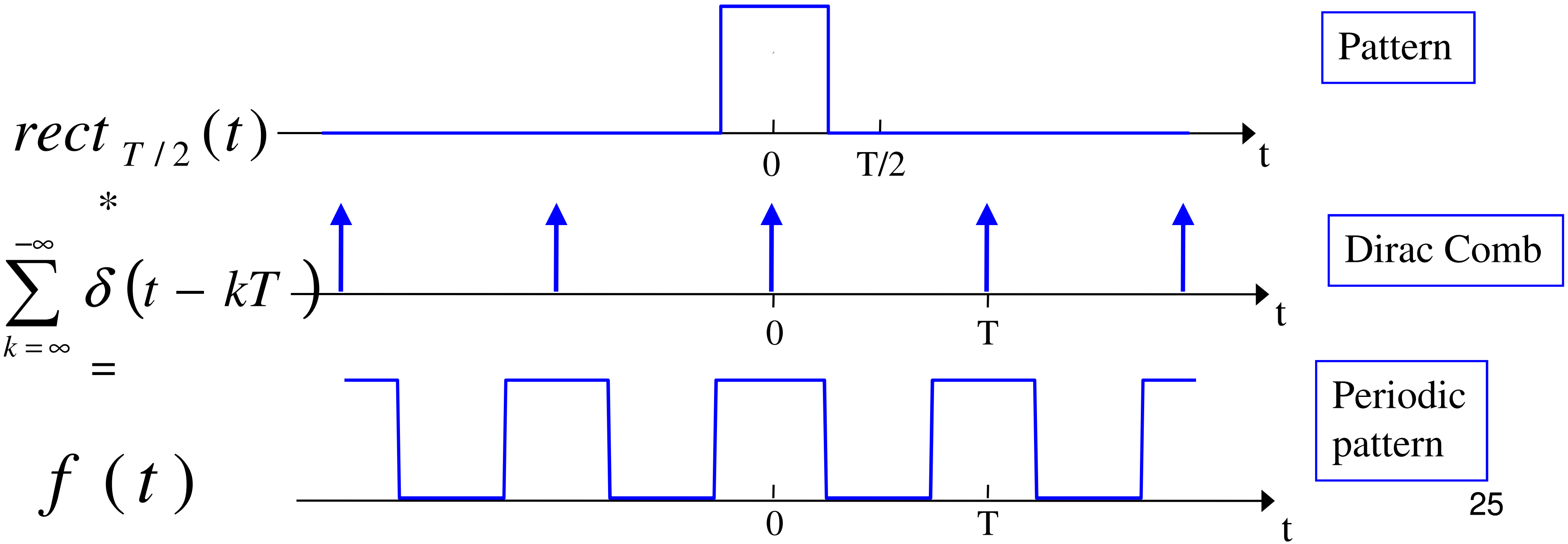
Dirac Delta function & Convolution

- Shifting property: $f(t) * \delta(t - t_0) = f(t - t_0)$

$$f(t) * \delta(t) = f(t)$$ (Dirac: neutral element for convolution)

- This property is used to repeat a pattern using convolution with a Dirac comb.

$$f(t) = rect_{T/2}(t) * \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} rect_{T/2}(t - kT)$$



LTI systems et Fourier Transform

- Let x(t) be the input to an LTI and y(t) be the output. Their respective Fourier Transform, X(f) et Y(f), are related by:

$$Y(f) = H(f)X(f)$$

where H(f) is the frequency response of the LTI. This is a complex number with a module and a phase.

- The frequency response is the Fourier Transform of the Impulse Response:

$$H(f) = TF\left[h(t)\right]$$

Property:

- The FT of a convolution product is the product of the FTs:

$$y(t) = h(t) * x(t) \xrightarrow{FT} Y(f) = H(f) \cdot X(f)$$

This is equivalent to: H(f)=FT(h(t)).

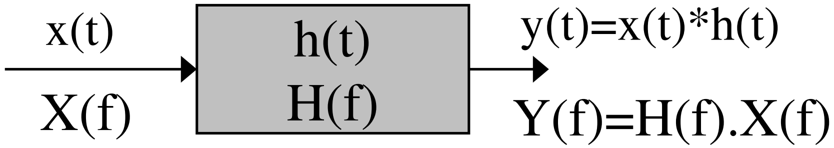
- The FT of a product is the convolution product of the FTs.

$$y(t) = w(t) \cdot x(t) \xrightarrow{FT} Y(f) = W(f) * X(f)$$

This property is used when a signal x(t) is windowed with a window w(t).

SUMMARY:

• A LTI can be characterized either by its impulse response or its frequency response:



$$H(f) = FT\left[h(t)\right]$$

• Filtering in the frequency space (multiplying by H(f)) is the same as convoluting in the time space (convoluting with h(t)).

Exercise: A LTI system has the following impulse response:

$$h(t) = \begin{cases} 1 & for \quad |t| \leq 1 \\ 0 & elsewhere \end{cases}$$

What is the fundamental problem with this response if one wants to realize it in practice?

Calculate the output y(t) for the following input:

$$x(t) = \begin{cases} 0 & for \quad t < 0 \\ 1 & for \quad t \geq 0 \end{cases}$$

III. Signal Energy, Correlation *(deterministic case)*

Signal having finite energy:

- Let x(t) be a signal in L^2. Its energy is by definition:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad < \infty$$

(finite energy signal)

Remark: this is a mathematical definition. The physical energy is obtained by multiplying by some factor.

- Its Energy Spectral Density (ESD) is a real positive quantity defined by:

$$S_{xx}(f) = |X(f)|^2$$

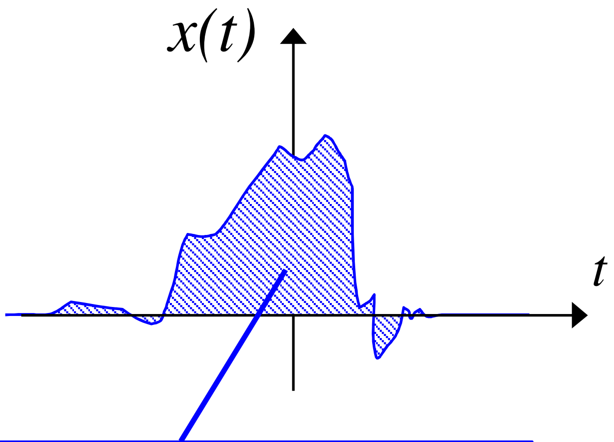
(finite energy signal)

- Parseval's relation states that energy can be calculated either in the time domain or in the frequency domain:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} S_{xx}(f) df$$

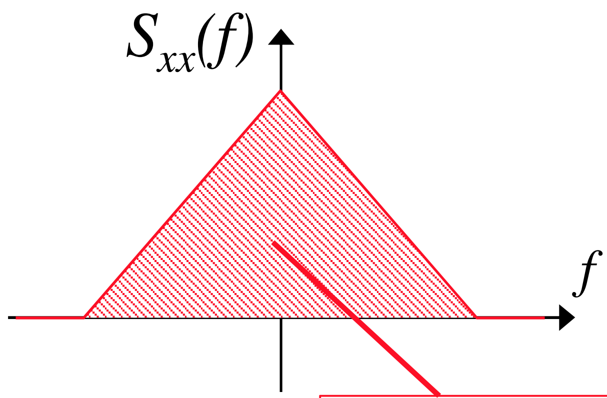
(finite energy signal)

TIME DOMAIN



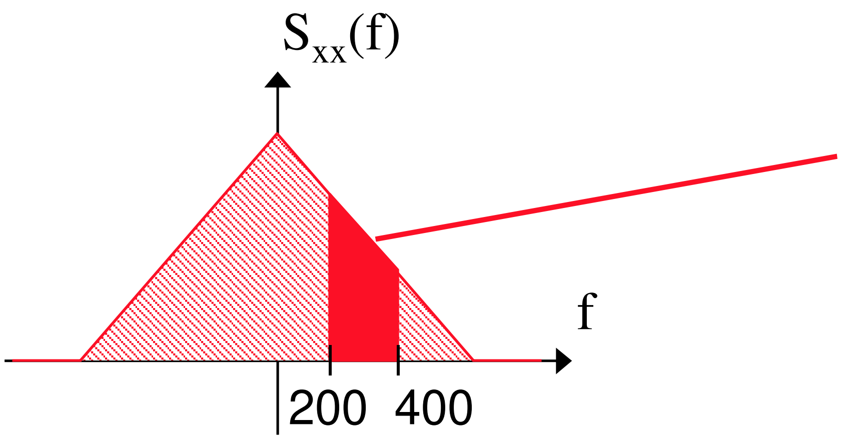
$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

FREQUENCY DOMAIN



$$E_x = \int_{-\infty}^{\infty} |X(f)|^2 df$$
$$= \int_{-\infty}^{\infty} S_{xx}(f) df$$

Energy in the signal between 200Hz and 400 Hz:



$$E_{[200-400]} = \int_{200}^{400} S_{xx}(f) df$$

Autocorrelation of a signal having finite energy

- Autocorrelation function (for a signal real having finite energy):

$$C_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt$$

$$C_{xx}(0) = E_x$$

(signal with finite energy)

The autocorrelation is maximal when the signal with a time offset τ resembles the signal.

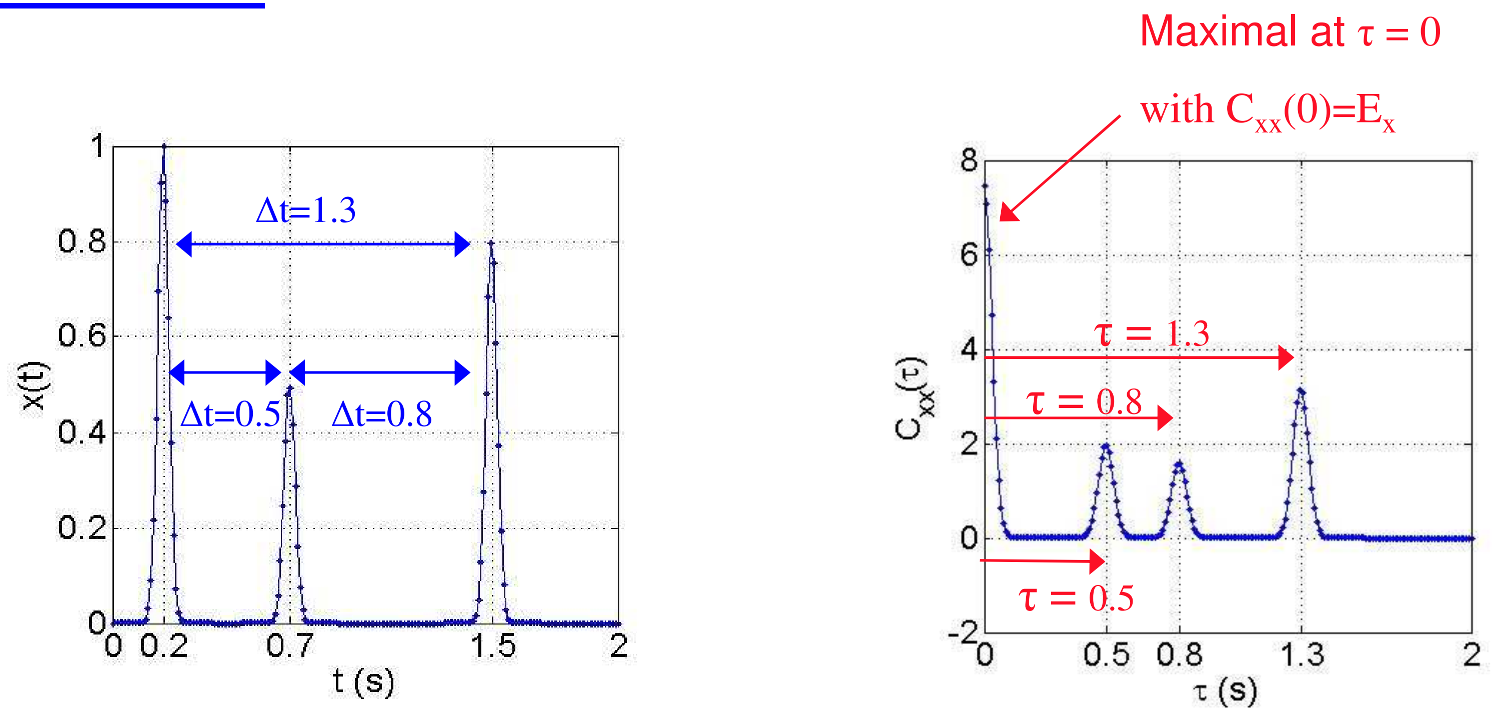
- «Wiener-Khintchine» theorem: « the energy spectral density is the Fourier Transform of the autocorrelation»:

$$S_{xx}(f) = \mathrm{FT}[C_{xx}(\tau)] = X^*(f)X(f) = |X(f)|^2$$

(signal with finite energy)

Remark: this theorem is usually known under this name for random processes (see lecture 2).

Auto-correlation



The autocorrelation for example allows to detect echoes in a signal.

Exercise:
What is the definition of the autocorrelation? Check that this is even. Consider the signal $x(t) = Rect_T(t)$. Calculate $C_{xx}(\tau)$. Calculate the Fourier Transform of $C_{xx}(\tau)$. What result do you recover? (recall what the Fourier of x(t) is RectT(t)).

Signal with finite power:

• For a stationnary signal (a sine for example), the energy is not finite (the signal is not in L^2), and one defines the mean power and the autocorrelation by:

$$P_x = \lim_{T \longrightarrow \infty} \frac{1}{T} \int_0^T |x(t)|^2 dt \quad < \infty \qquad \text{(while } E_x = \infty\text{)}$$

$$C_{xx}(\tau) = \lim_{T \longrightarrow \infty} \frac{1}{T} \int_0^T x(t)\, x(t+\tau)\, dt \qquad C_{xx}(0) = P_x$$

(signal with finite power)

• **Power Spectral Density (PSD):** $S_{xx}(f) = FT[C_{xx}(\tau)]$

(signal with finite power)

• The PSD is also given by:

(signal with finite power)

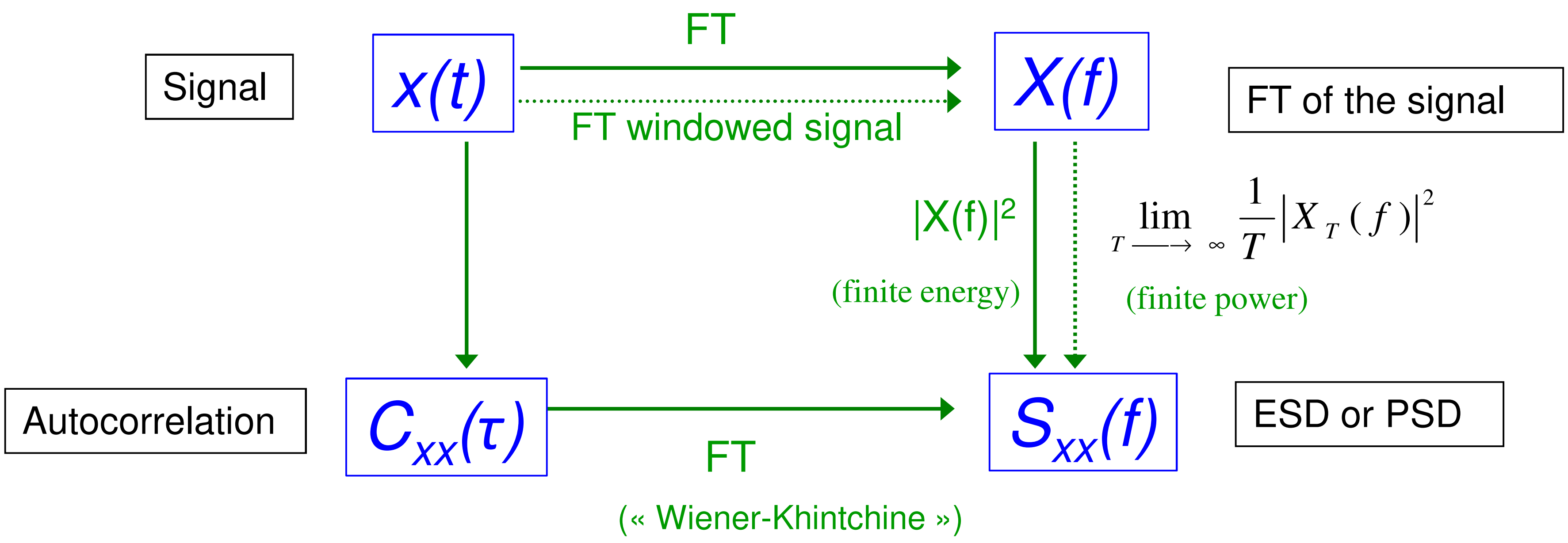
$$S_{xx}(f) = \lim_{T \longrightarrow \infty} \frac{1}{T} X_T(f)^* X_T(f) = \lim_{T \longrightarrow \infty} \frac{1}{T} |X_T(f)|^2$$

where $X_T(f) = FT\,[x(t) \cdot rect_T(t)] \qquad \neq X^*(f) X(f)$

The PSD is defined by taking a limit, using the Fourier Transform of the windowed signal $x(t).rect_T(t)$. The limit is taken for a window of larger and larger width T.

SUMMARY:

• Two ways for calculating the ESD/PSD of a signal:



Remark: for now we are dealing with deterministic signals. We will extend this to random signals in lecture 3.